Mathematical Induction

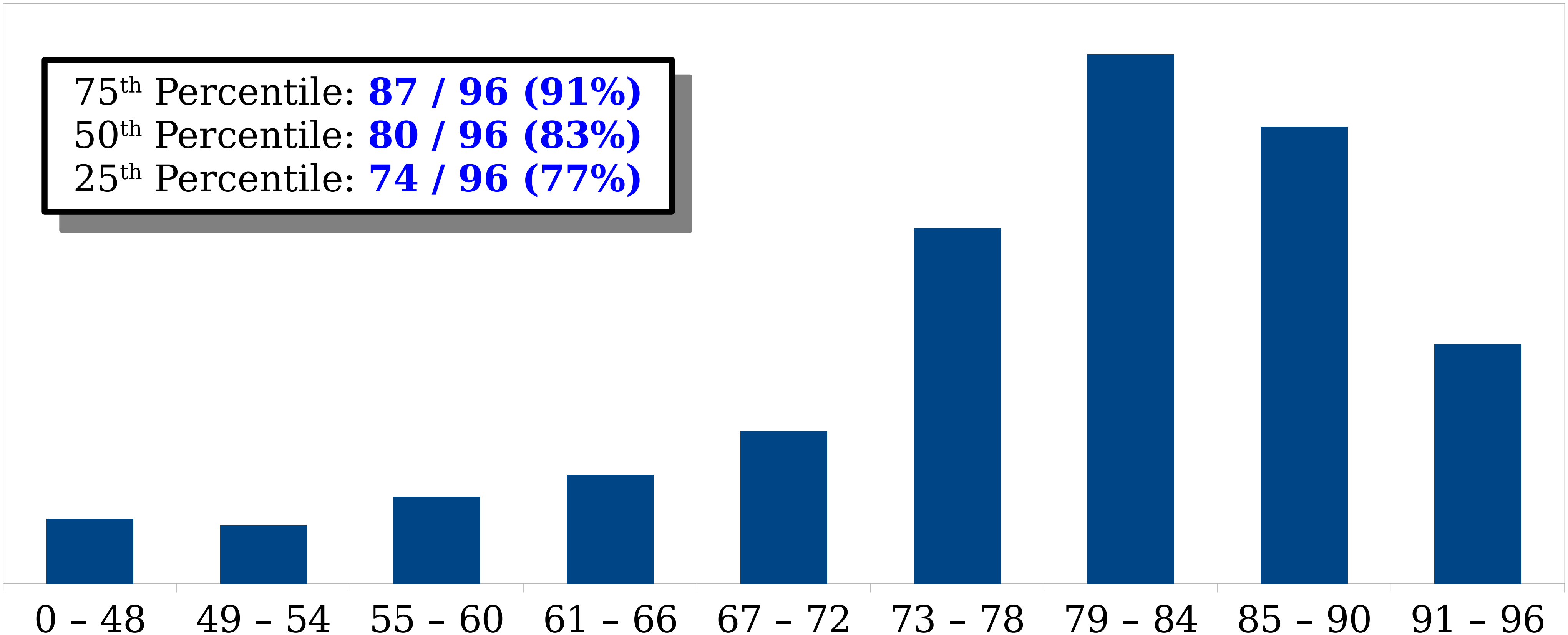
Part One

Quick Announcements!

Problem Set Four

- Problem Set Four is due this Friday at 1:00PM.
  - It's smaller than usual.
- PS3 grades and solutions are posted!
- ***Recommendation:*** As soon as you can, review all the feedback you got on PS3 and ask yourself these questions:
  - Based on the proofwriting and style feedback you received, do you know what specific changes you'd make to your answers?
  - If you made any logic errors, do you understand what those errors are to the point that you could explain them to someone else?
- Feel free to stop by office hours or to visit EdStem if you have questions. We're happy to help out! You can do this!
- Exam grading is this Sunday.

Problem Set Three Graded



Okay, let’s kick off our exploration of today’s material with some kinetic activity.

Let’s do the wave!

The Wave

- If done properly, everyone will eventually end up joining in.
- Why is that? There are two primary components:
  - Someone (me!) started everyone off.
  - Once the person before you did the wave, you did the wave.

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$$\boldsymbol{P(0)} \textbf{ is true}$$

…and it stays true…

and

$$\forall k \in \mathbb{N}.\ (P(k) \rightarrow P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

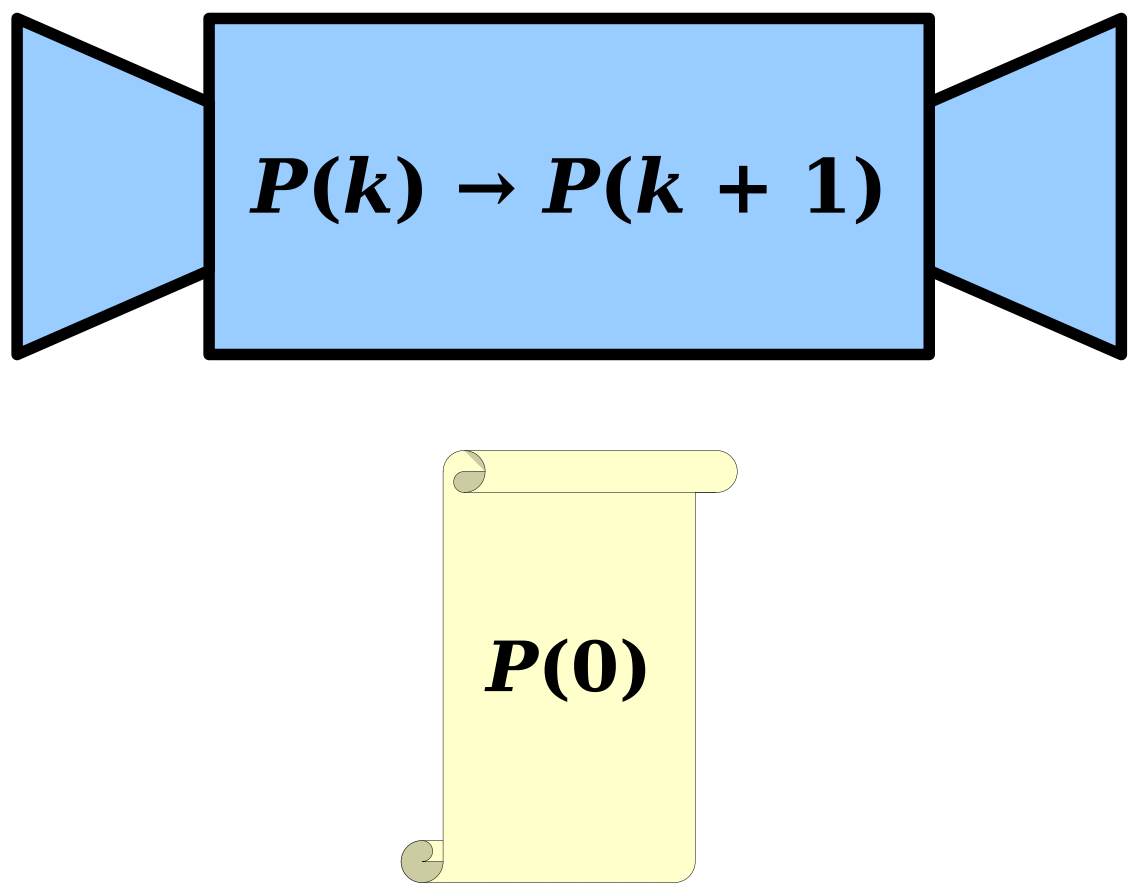
Induction, Intuitively

$$\boldsymbol{P(0)}$$

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

- It's true for 0.
- Since it's true for 0, it's true for 1.
- Since it's true for 1, it's true for 2.
- Since it's true for 2, it's true for 3.
- Since it's true for 3, it's true for 4.
- Since it's true for 4, it's true for 5.
- Since it's true for 5, it's true for 6.
- …

Why Induction Works



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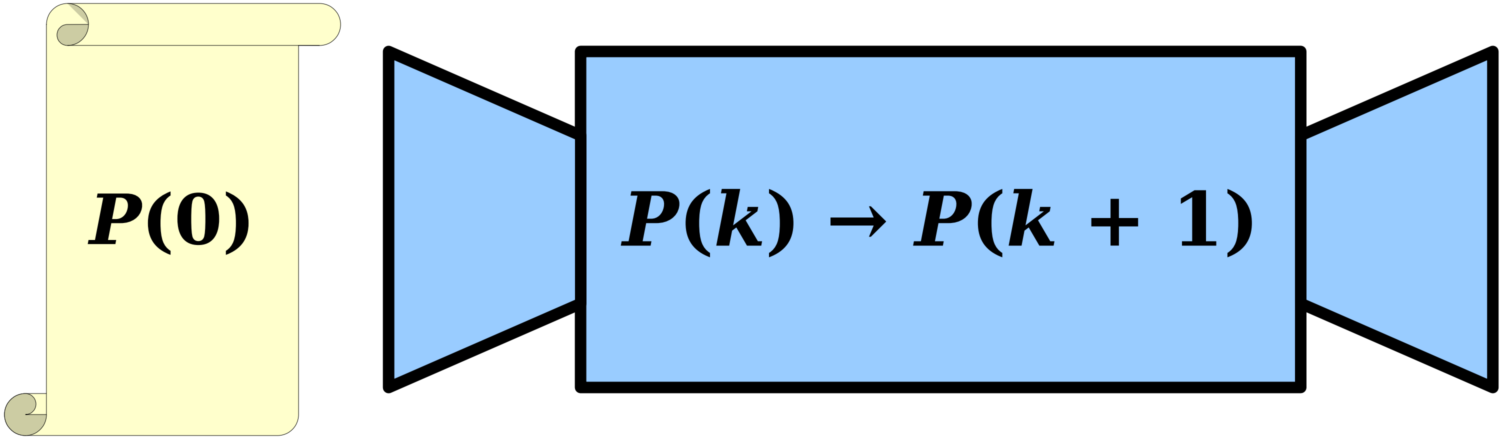

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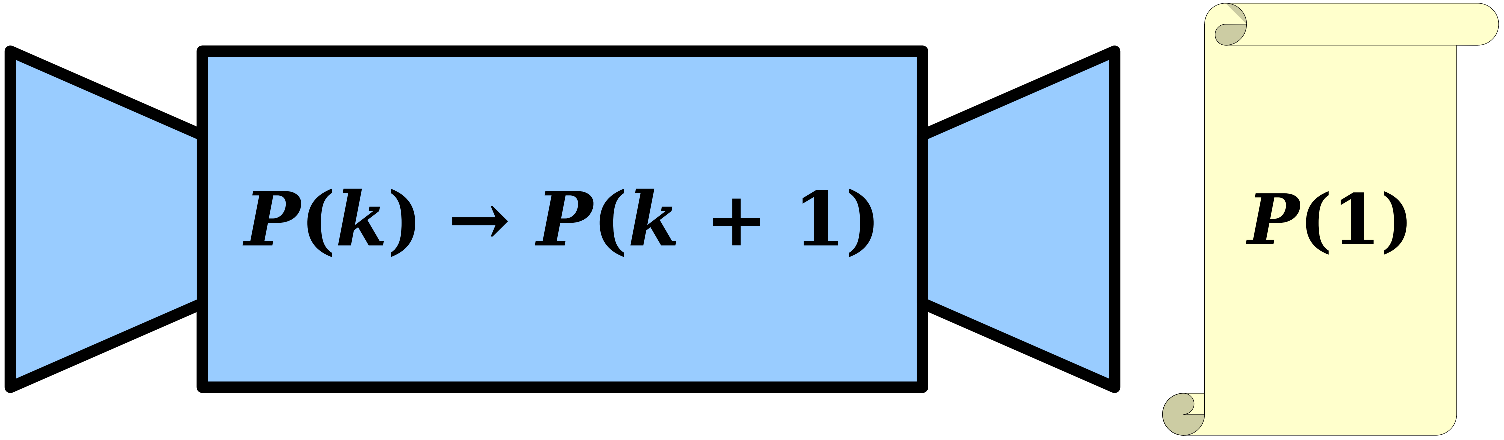

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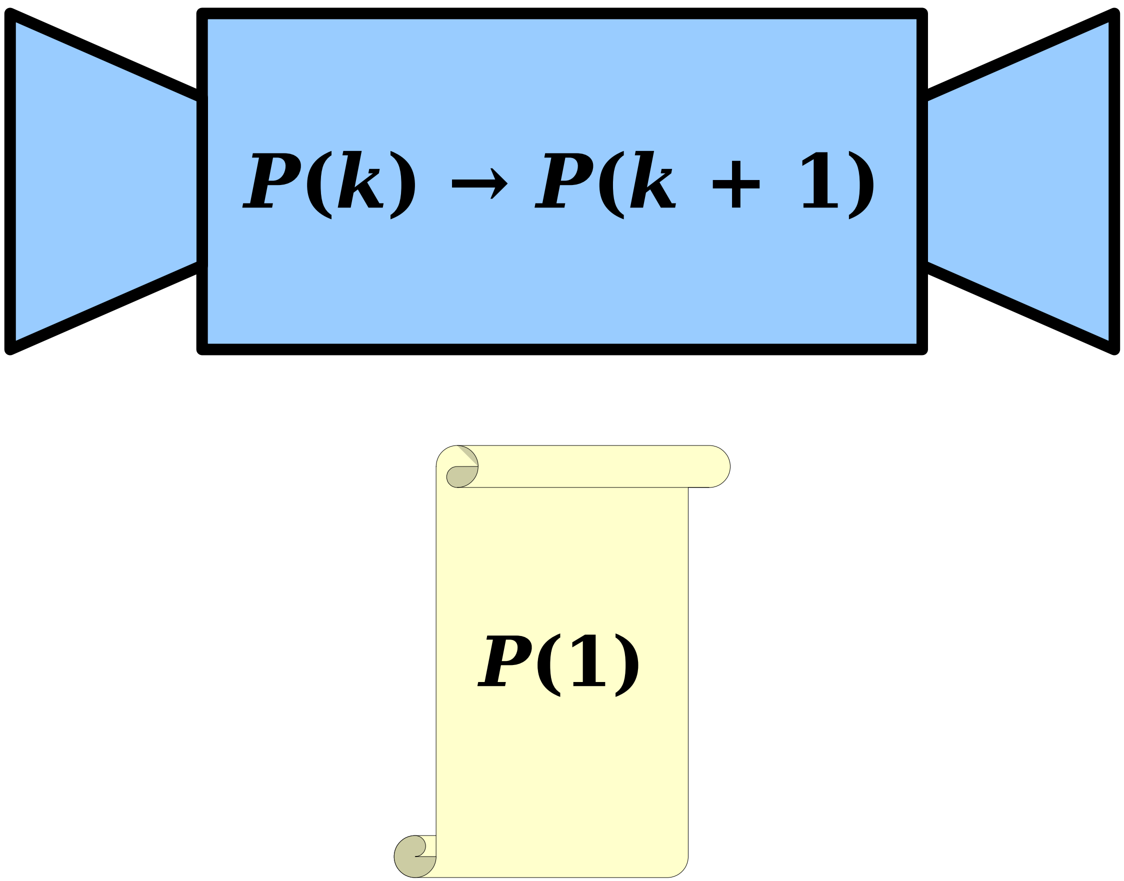

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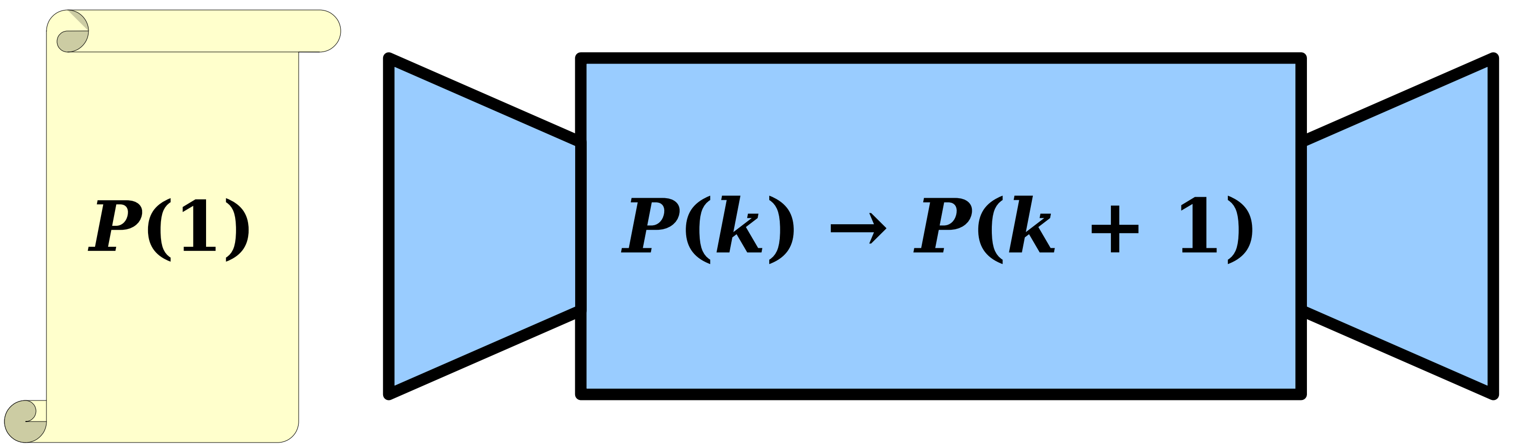

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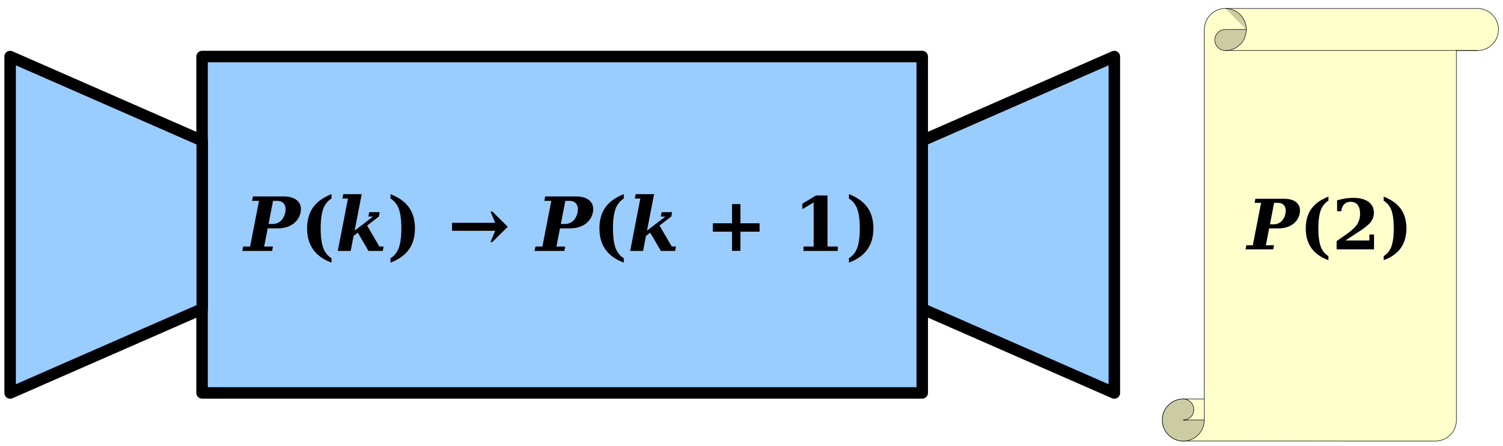

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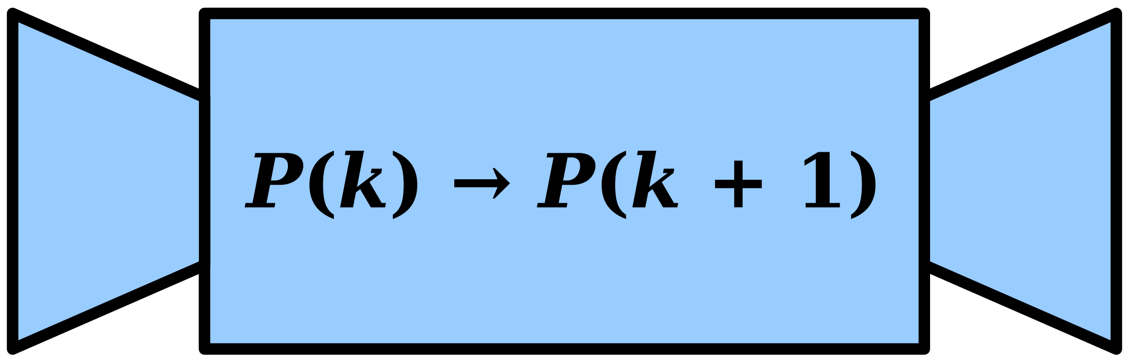

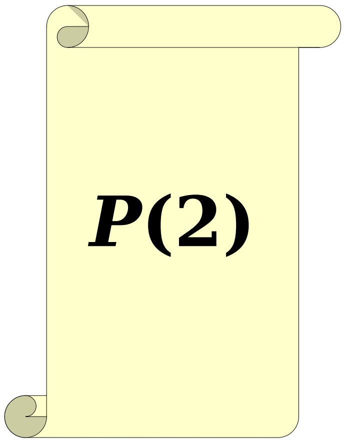

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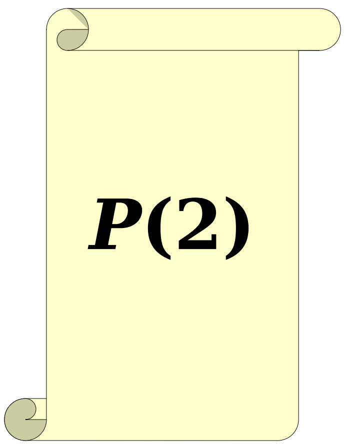

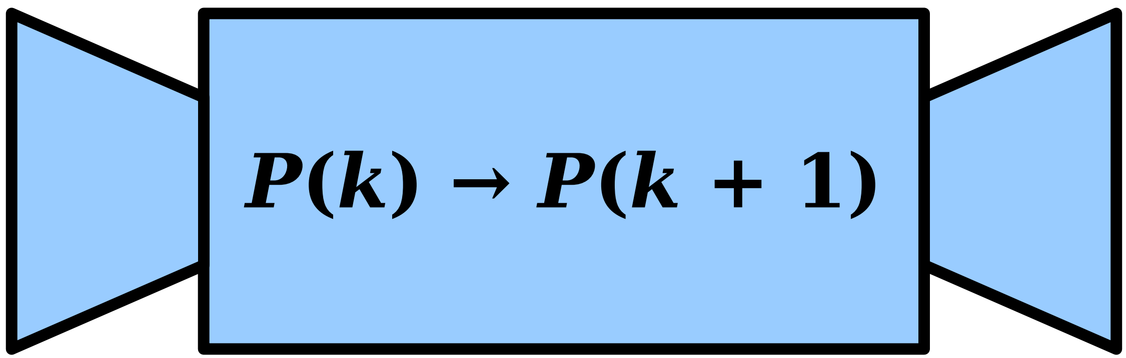

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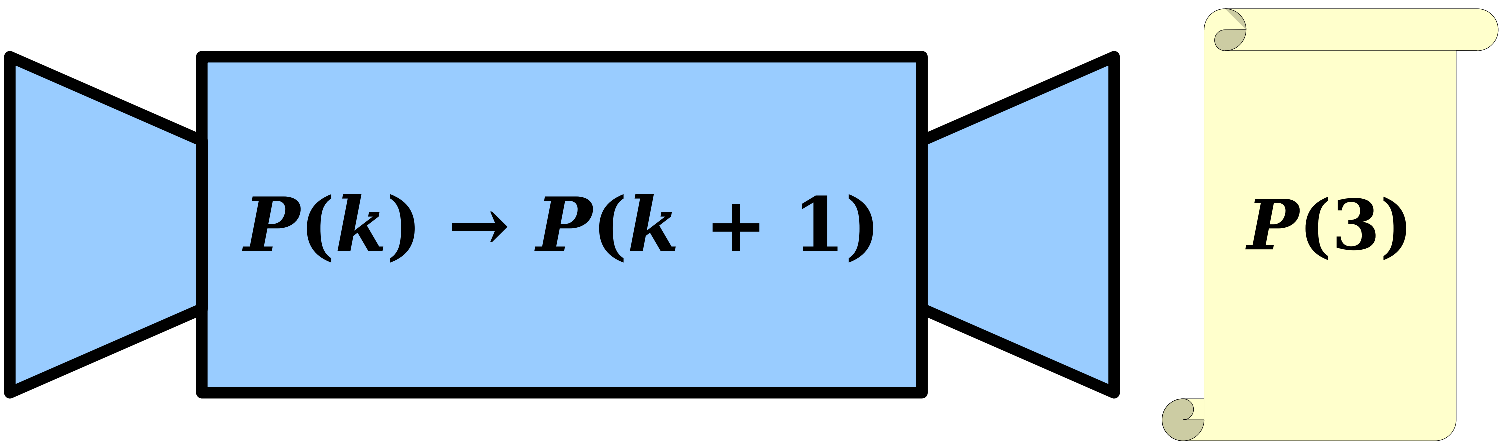

Proof by Induction

- A ***proof by induction*** is a way to use the principle of mathematical induction to show that some result is true for all natural numbers n.
- In a proof by induction, there are three steps:
  - Prove that $P(0)$ is true.
    - This is called the ***basis*** or the ***base case***.
  - Prove that if $P(k)$ is true, then $P(k+1)$ is true.
    - This is called the ***inductive step***.
    - The assumption that $P(k)$ is true is called the ***inductive hypothesis***.
  - Conclude, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$.

Some Sums

$$2^0$$

$$2^0 + 2^1$$

$$2^0 + 2^1 + 2^2$$

$$2^0 + 2^1 + 2^2 + 2^3$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4$$

$$\mathbf{2^0} = 1$$

$$\mathbf{2^0 + 2^1} = 1 + 2 = 3$$

$$\mathbf{2^0 + 2^1 + 2^2} = 1 + 2 + 4 = 7$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3} = 1 + 2 + 4 + 8 = 15$$

$$\mathbf{2^0 + 2^1 + 2^2 + 2^3 + 2^4} = 1 + 2 + 4 + 8 + 16 = 31$$

$$2^0 = 1 \quad = 2^1 - 1$$

$$2^0 + 2^1 = 1 + 2 = 3 = 2^2 - 1$$

$$2^0 + 2^1 + 2^2 = 1 + 2 + 4 = 7 = 2^3 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 = 1 + 2 + 4 + 8 = 15 = 2^4 - 1$$

$$2^0 + 2^1 + 2^2 + 2^3 + 2^4 = 1 + 2 + 4 + 8 + 16 = 31 = 2^5 - 1$$

Theorem: The sum of the first n powers of two is $2^n - 1$.

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what P(0) actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

So we ask the reader to pick some k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state P(k+1), which is what we want to prove. Now, we can use any proof technique we want to prove it.

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For ou … meaning that th … – 1. Since the su … d $2^0 - 1$ is zero …

Here, we'll use our inductive hypothesis (the assumption that $P(k)$ is true) to simplify a complex expression. This is a common theme in inductive proofs.

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Therefore, $P(k + 1)$ is true, completing the induction.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1 \qquad (1)$$

In a proof by induction, we need to prove that

✓ $P(0)$ is true

If $P(k)$ is true, then $P(k+1)$ is true.

We n … um

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notic…

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$$2^0 + 2^1 + \dots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \dots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \dots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

A Quick Aside

- This result helps explain the range of numbers that can be stored in an `int`.
- If you have an unsigned 32-bit integer, the largest value you can store is given by $1 + 2 + 4 + 8 + \ldots + 2^{31} = 2^{32} - 1$.
- This formula for sums of powers of two has many other uses as well. You'll see one on Friday.

Structuring a Proof by Induction

- Define some predicate P that you'll show, by induction, is true for all natural numbers.
- Prove the base case:
  - State that you're going to prove that $P(0)$ is true, then go prove it.
- Prove the inductive step:
  - Say that you're assuming $P(k)$ for some arbitrary natural number k, then write out exactly what that means.
  - Say that you're going to prove $P(k+1)$, then write out exactly what that means.
  - Prove that $P(k+1)$ using any proof technique you'd like!
- This is a rather verbose way of writing inductive proofs. As we get more experience with induction, we'll start leaving out some details from our proofs.

The Counterfeit Coin Problem

Problem Statement

- You are given a set of three seemingly identical coins, two of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only one weighing on the balance, find the counterfeit coin.

How?

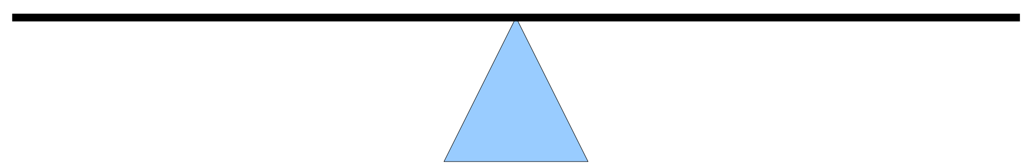
Answer at
https://cs103.stanford.edu/pollev

Finding the Counterfeit Coin

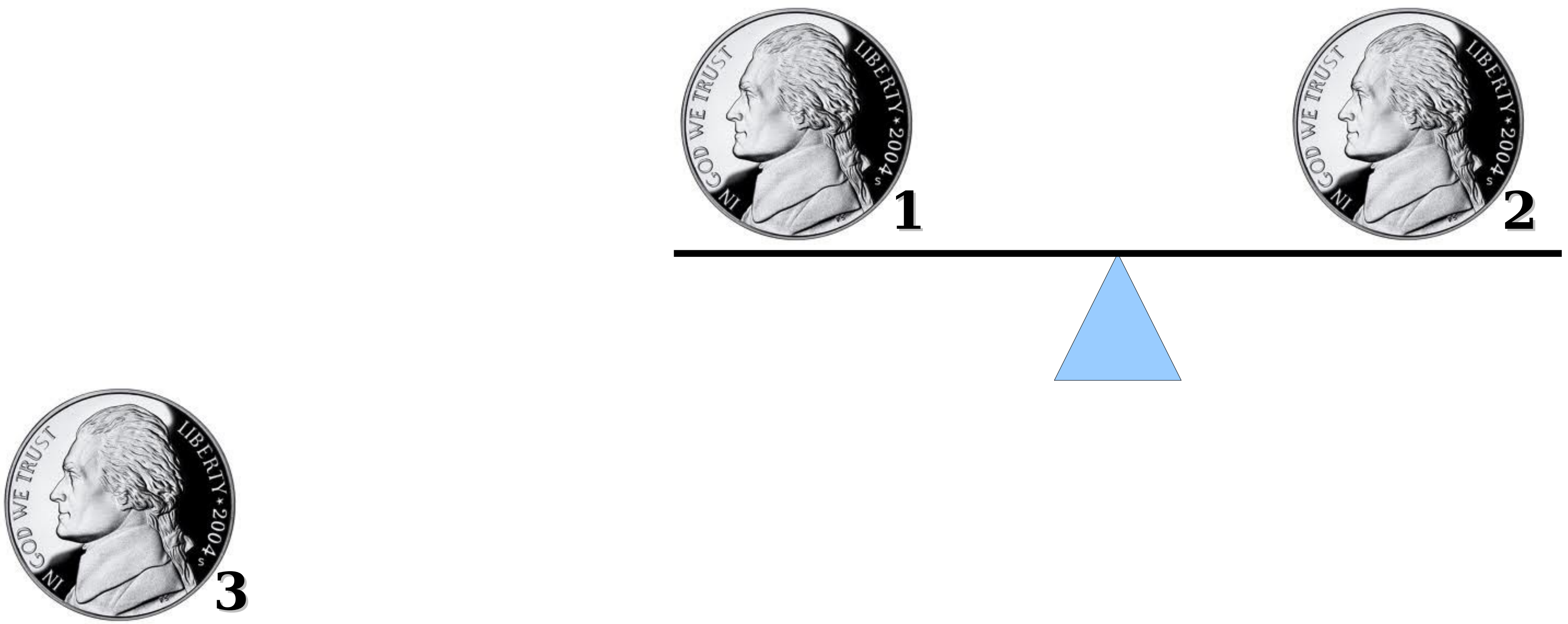
1

2

3



Finding the Counterfeit Coin



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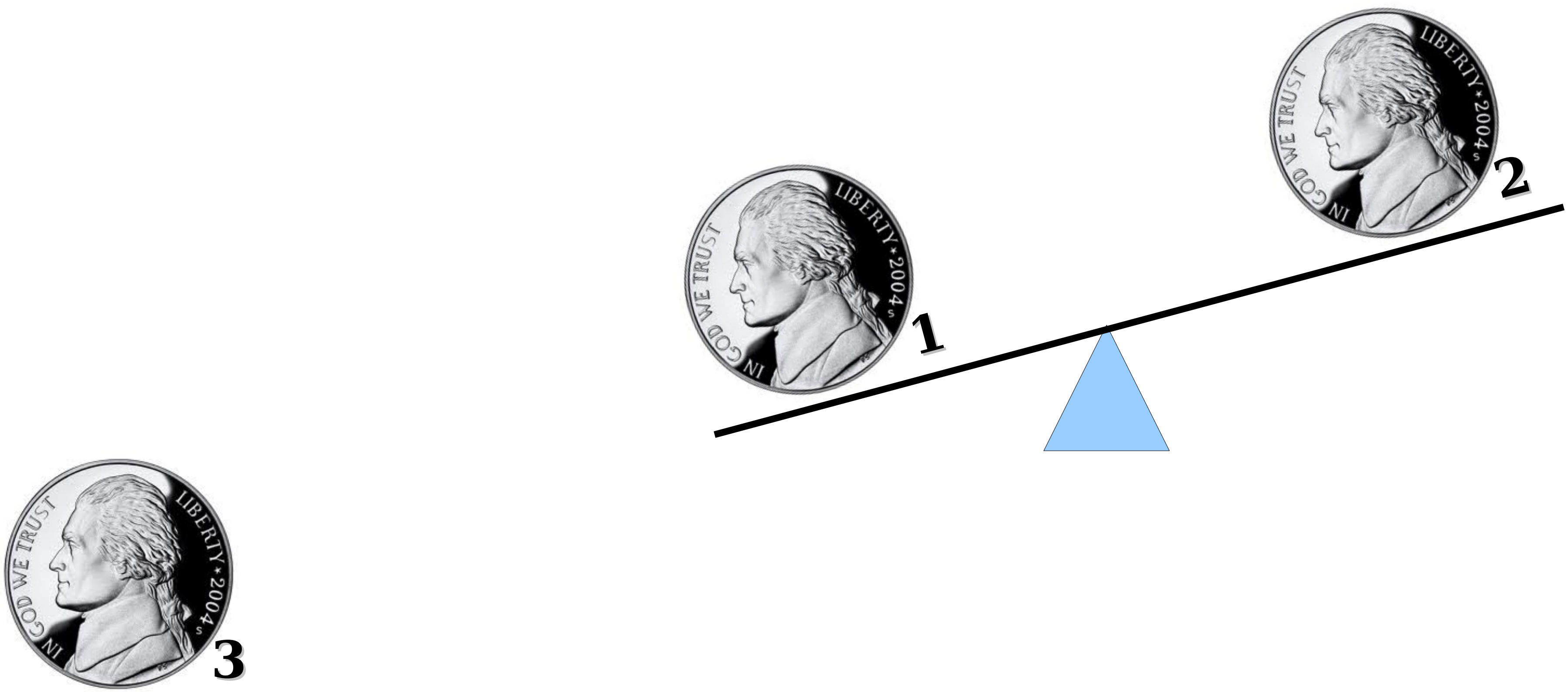

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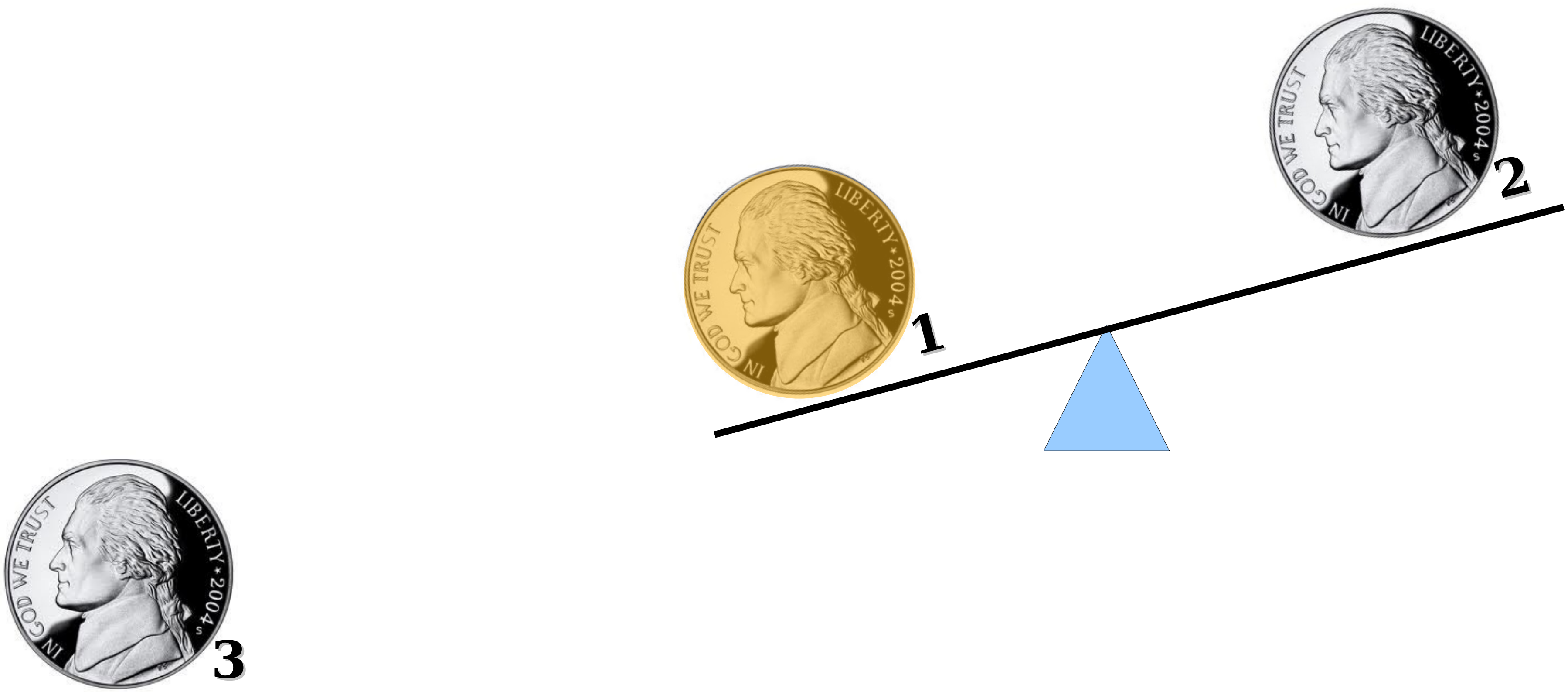

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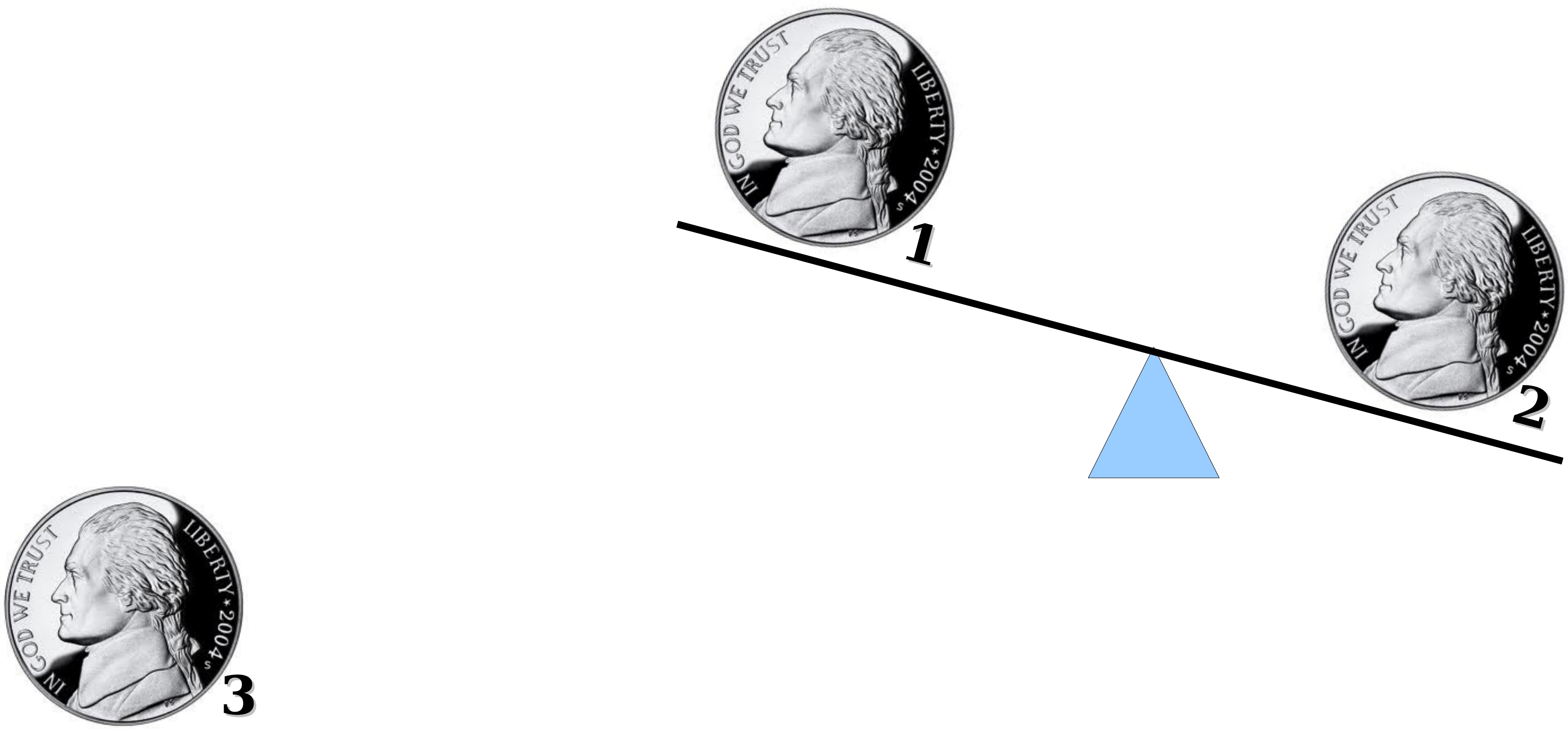

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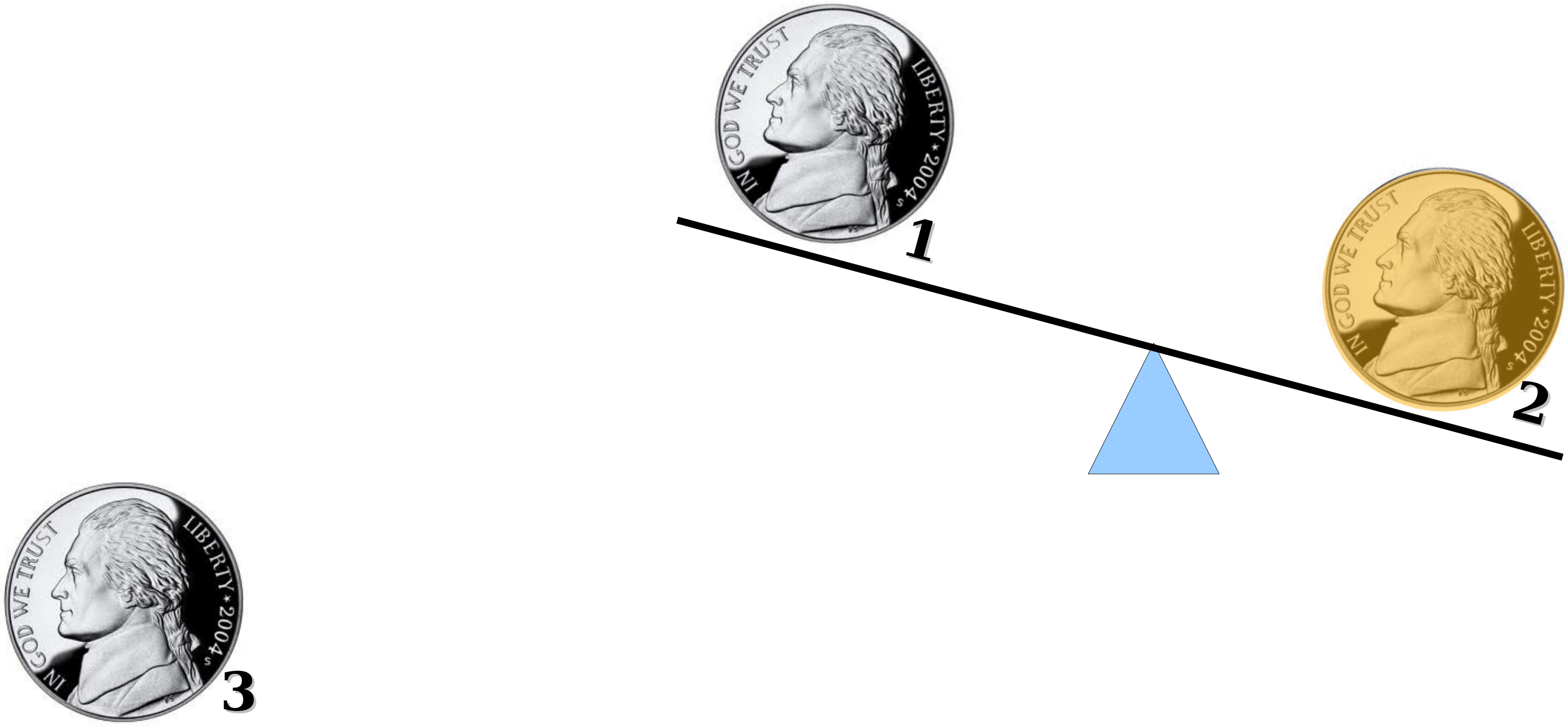

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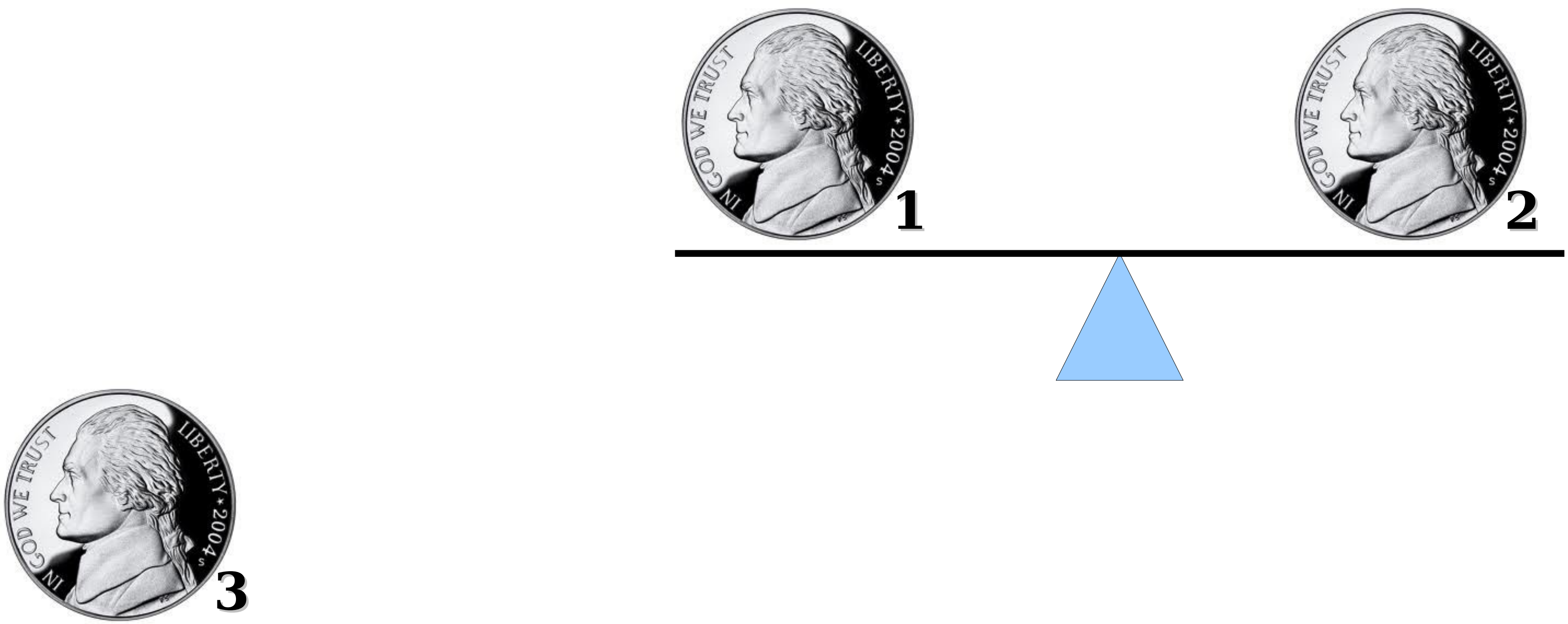

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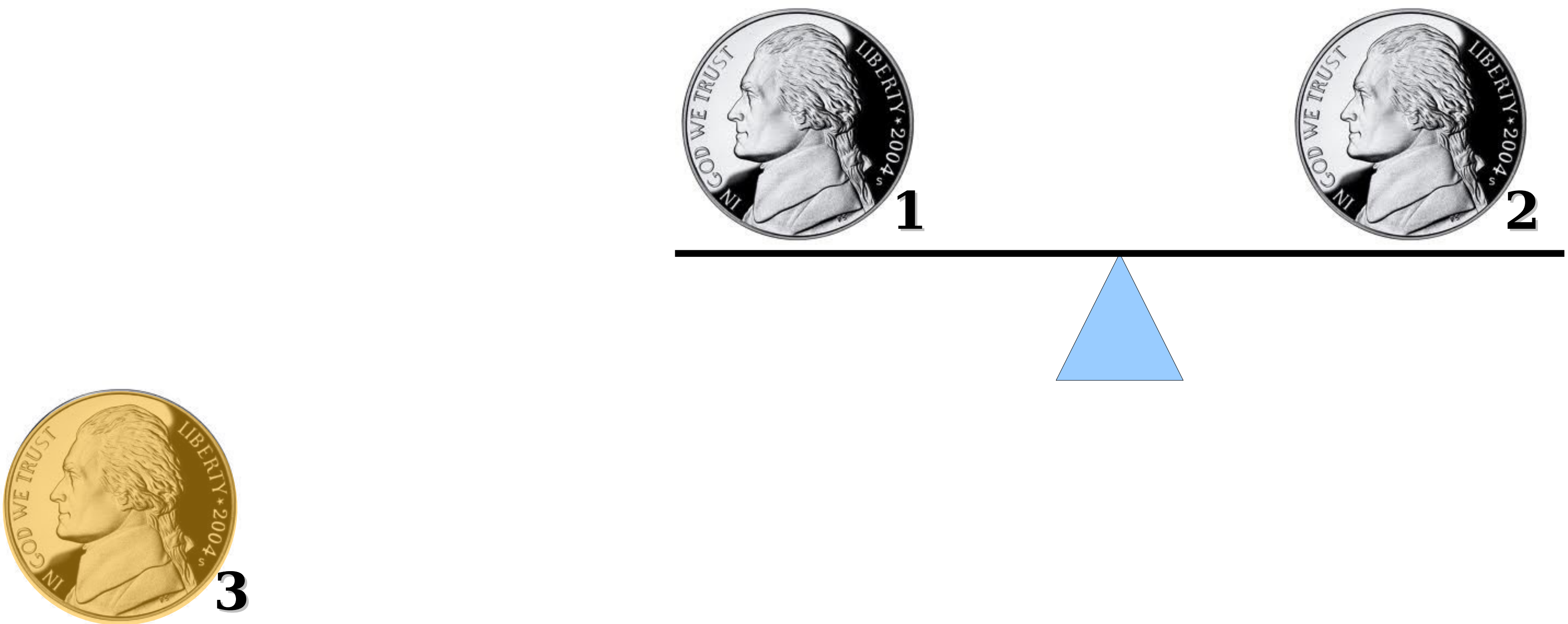
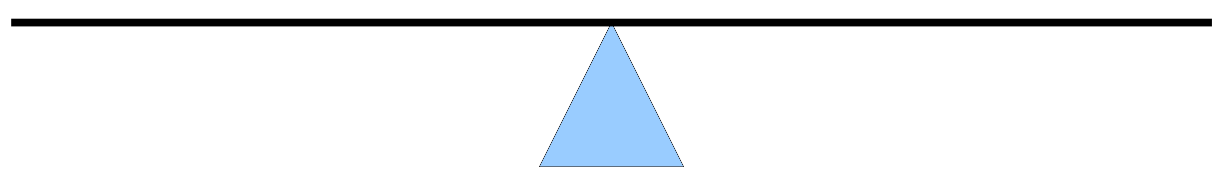


A Harder Problem

- You are given a set of ***nine*** seemingly identical coins, eight of which are real and one of which is counterfeit.
- The counterfeit coin weighs more than the rest of the coins.
- You are given a balance. Using only ***two*** weighings on the balance, find the counterfeit coin.

Finding the Counterfeit Coin



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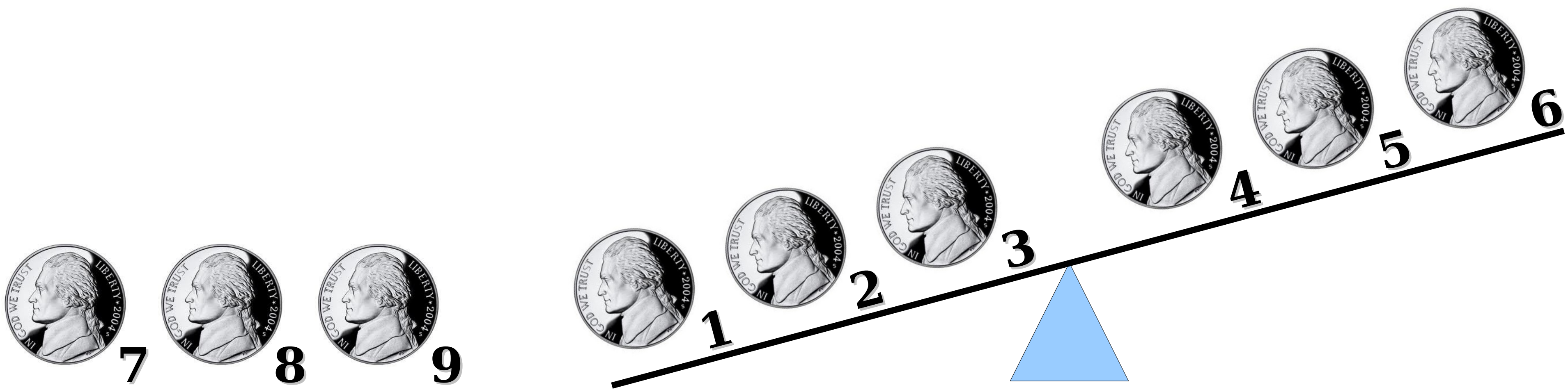

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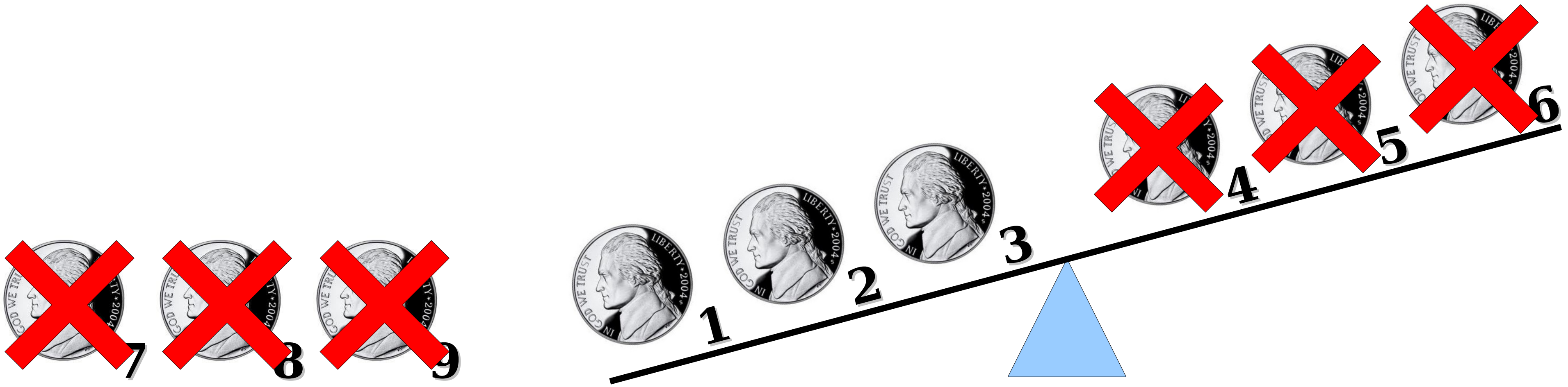
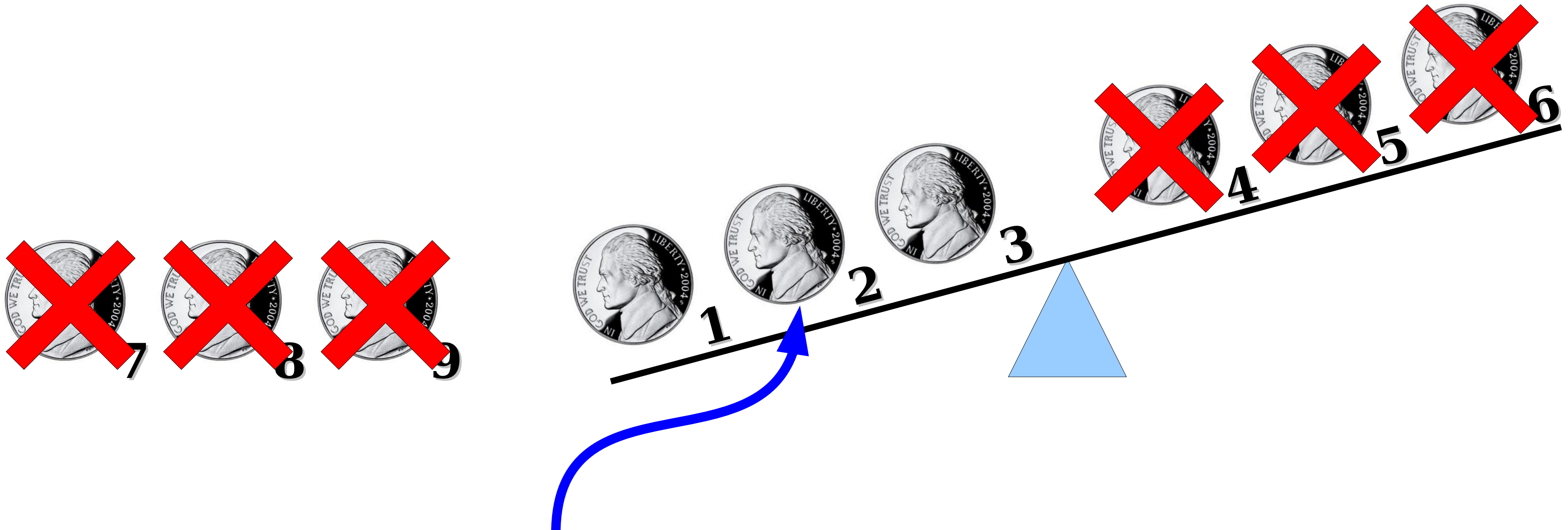
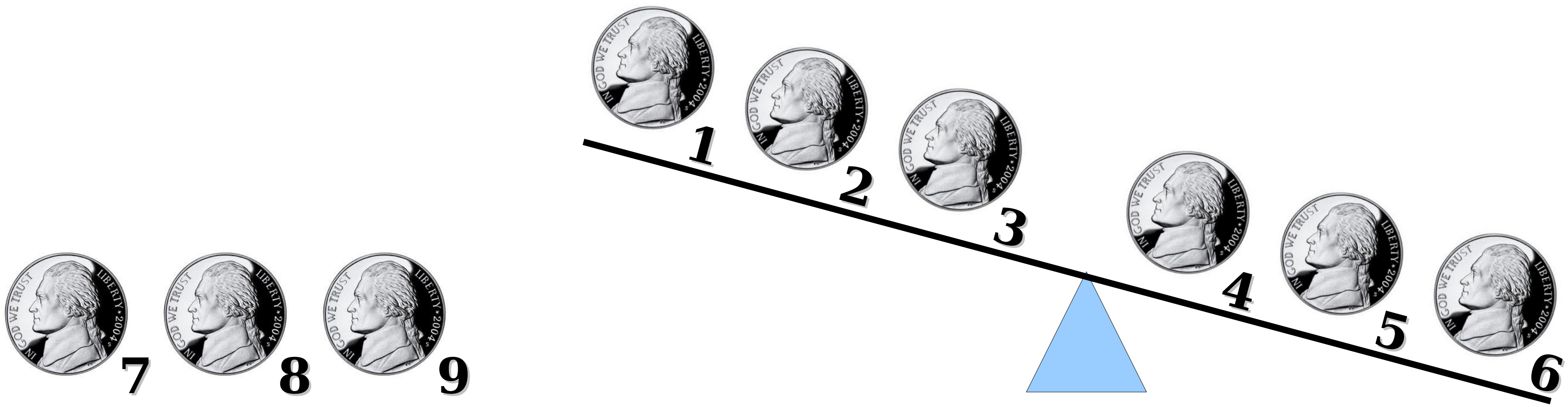


Finding the Counterfeit Coin



Now we have one weighing to find the counterfeit out of these three coins.

Finding the Counterfeit Coin



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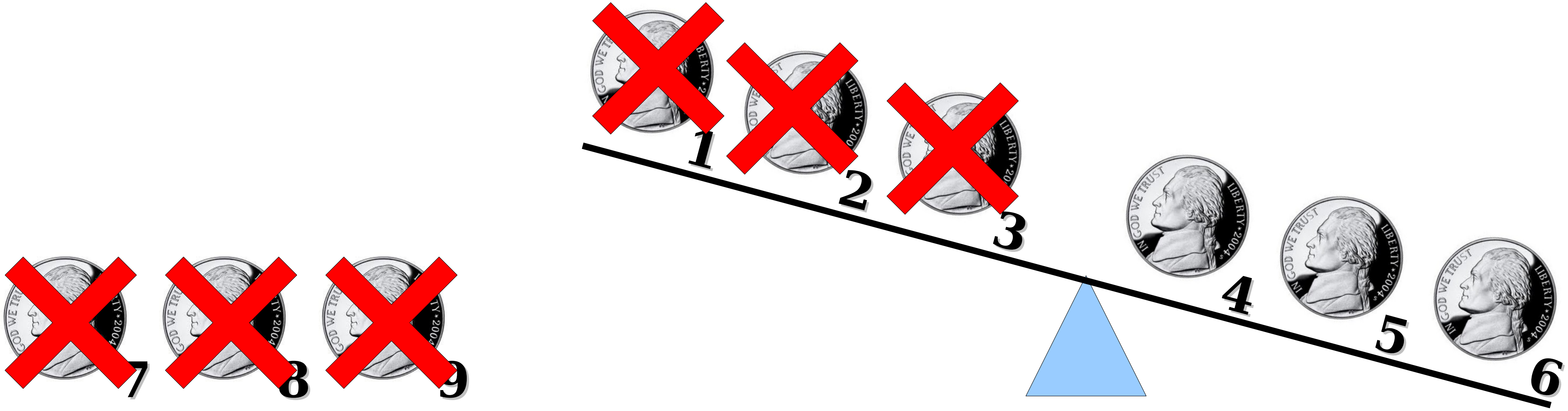

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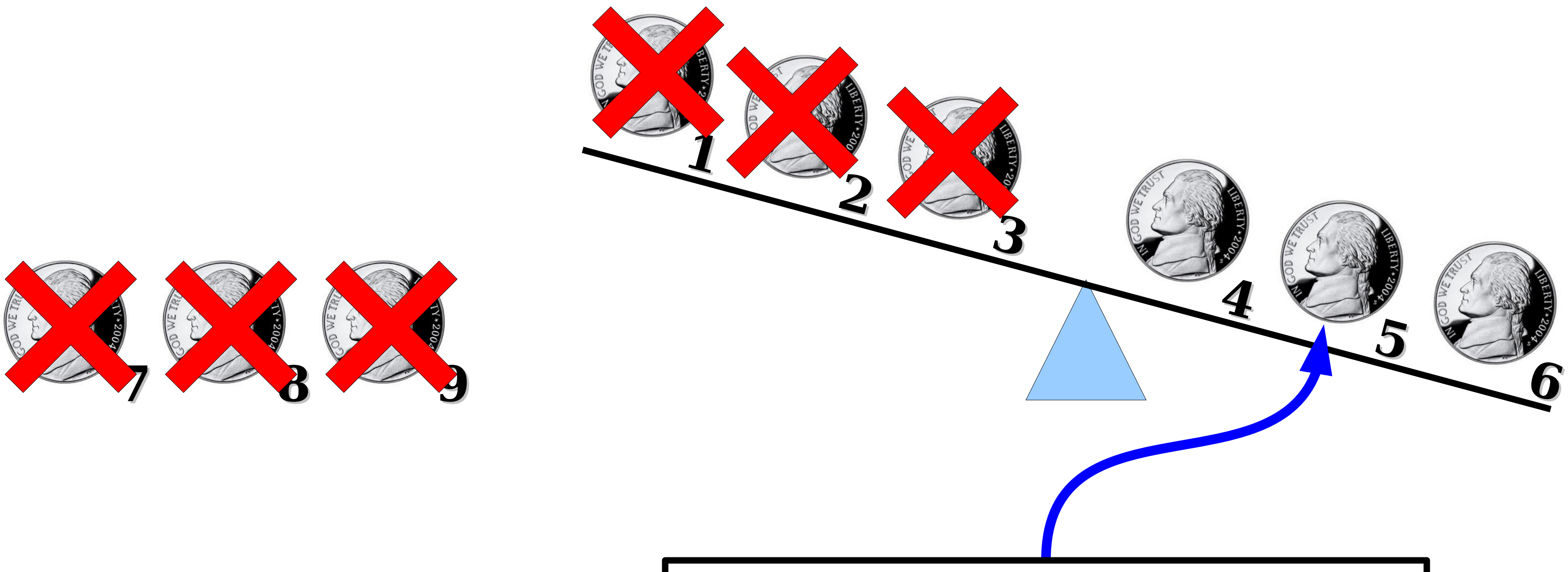

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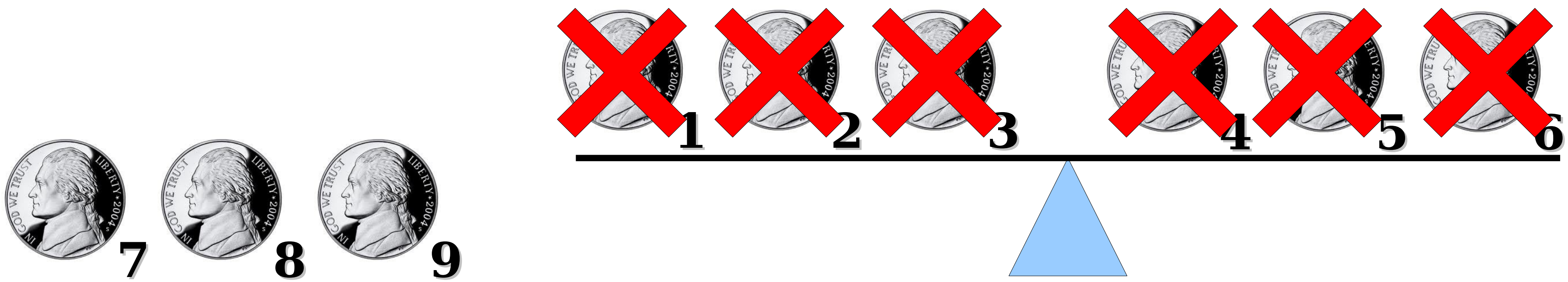

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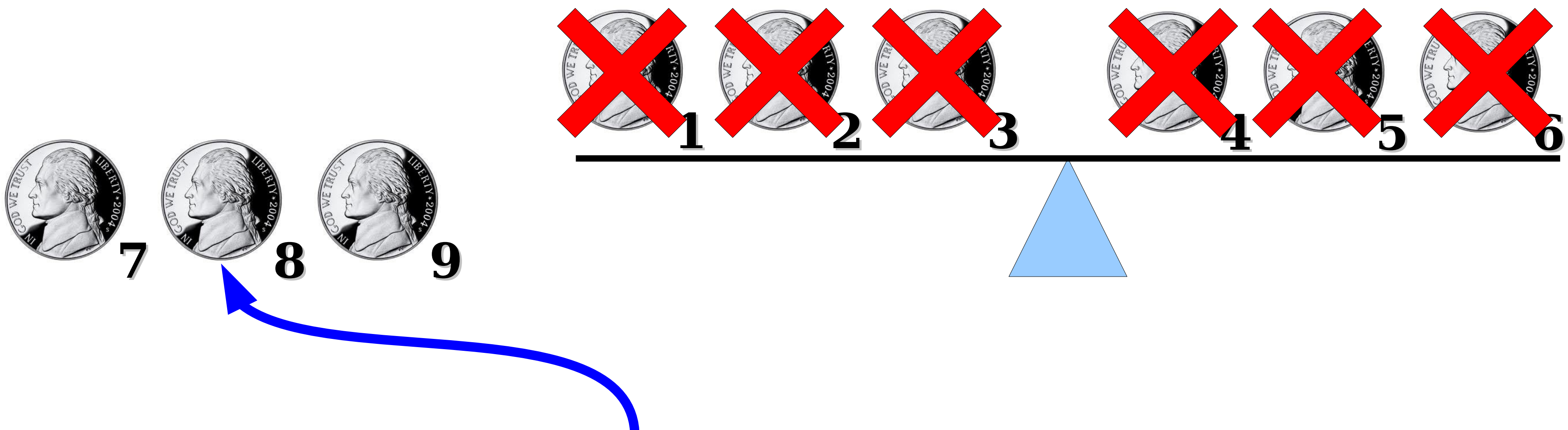

Now we have one weighing to find the counterfeit out of these three coins.

Can we generalize this?

A Pattern

- Assume out of the coins that are given, exactly one is counterfeit and weighs more than the other coins.
- If we have no weighings, how many coins can we have while still being able to find the counterfeit?
  - ***One*** coin, since that coin has to be the counterfeit!
- If we have one weighing, we can find the counterfeit out of ***three*** coins.
- If we have two weighings, we can find the counterfeit out of ***nine*** coins.

So far, we have

$1, 3, 9 = 3^0, 3^1, 3^2$

Does this pattern continue?

Theorem: If exactly one coin in a group of 3^n coins is heavier than the rest, that coin can be found using only n weighings on a balance.

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Proof: Let $P(n)$ be the following statement:

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At the start of the proof, we tell the reader what predicate we're going to show is true for all natural numbers n, then tell them we're going to prove it by induction.

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Here, we state what $P(0)$ actually says. Now, can go prove this using any proof techniques we'd like!

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The goal of this step is to prove

"If $P(k)$ is true, then $P(k+1)$ is true."

So we ask the reader to choose an arbitrary k, assume that $P(k)$ is true, then try to prove $P(k+1)$.

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Here, we explicitly state $P(k+1)$, which is what we want to prove. Now, we can use any proof technique we want to try to prove it.

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Suppose we have 3^{k+1} coins with one heavier than the others.

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If exact … rest,
that c … ce.

We'll use i … om which the theore …

Here, we use our inductive hypothesis (the assumption that $P(k)$ is true) to solve this simpler version of the overall problem.

As our bas … we have a set of 3^0 … find that coin with zero weighings. This is true because if we have just one coin, it's vacuously heavier than all the others, and no weighings are needed.

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For the … so we can … : that we …

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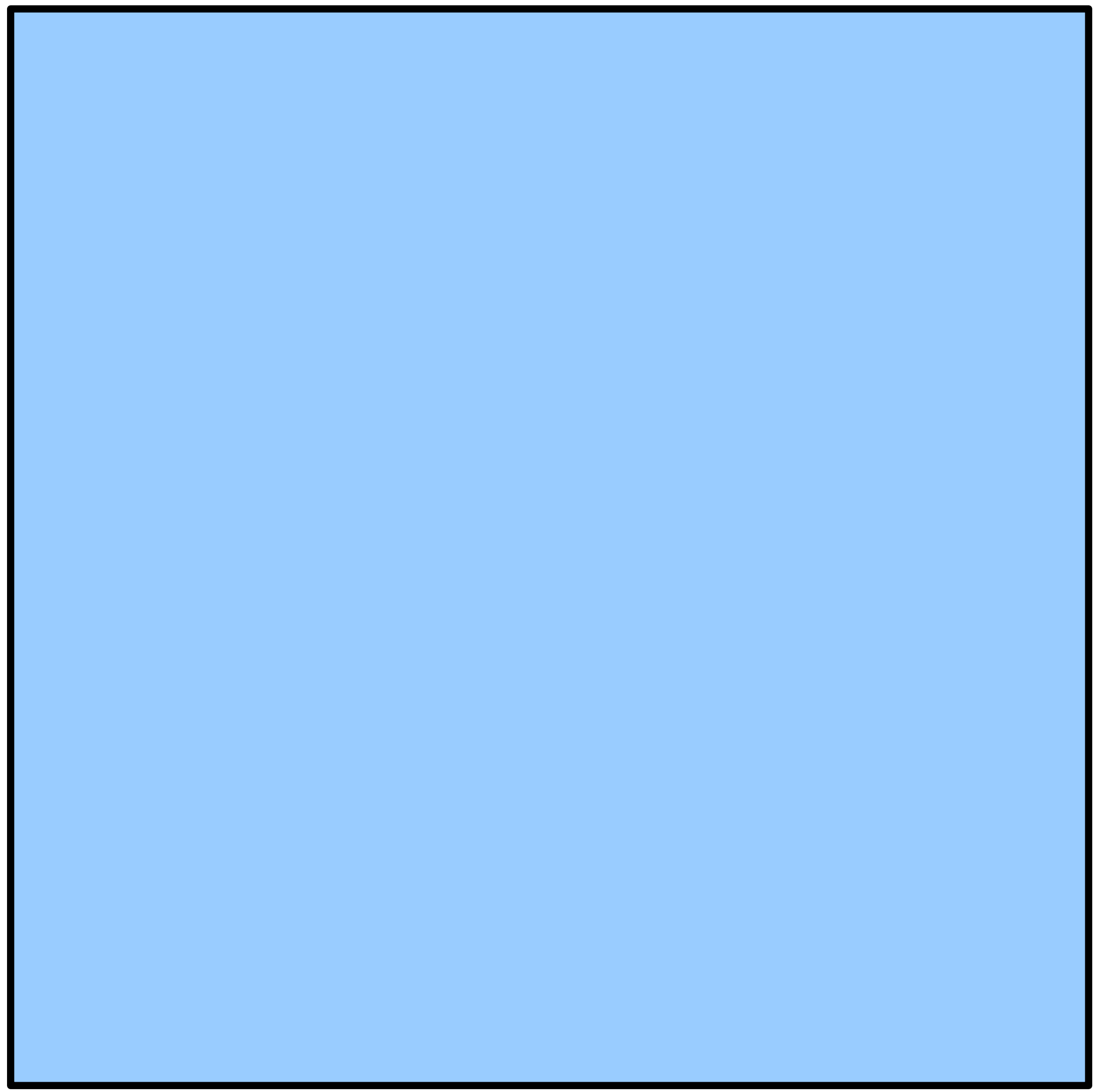
Some Fun Problems

- Here's some nifty variants of this problem that you can work through:
  - Suppose that you have a group of coins where there's either exactly one heavier coin, or all coins weigh the same amount. If you only get k weighings, what's the largest number of coins where you can find the counterfeit or determine none exists?
  - What happens if the counterfeit can be either heavier or lighter than the other coins? What's the maximum number of coins where you can find the counterfeit if you have k weighings?
  - Can you find the counterfeit out of a group of more than 3^k coins with k weighings?
  - Can you find the counterfeit out of any group of at most 3^k coins with k weighings?

Quirky Interludes

Variations on Induction

Subdividing a Square



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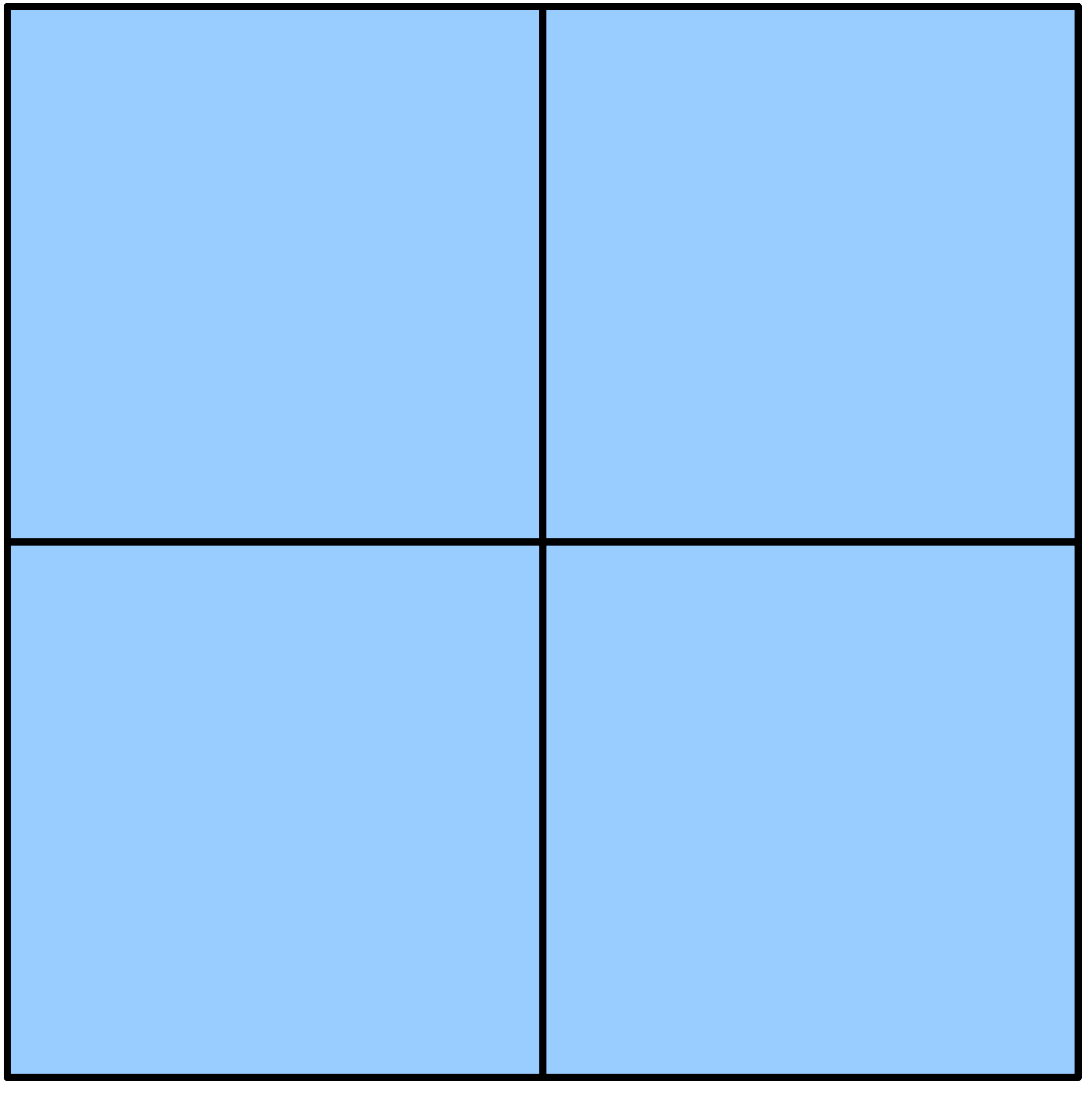
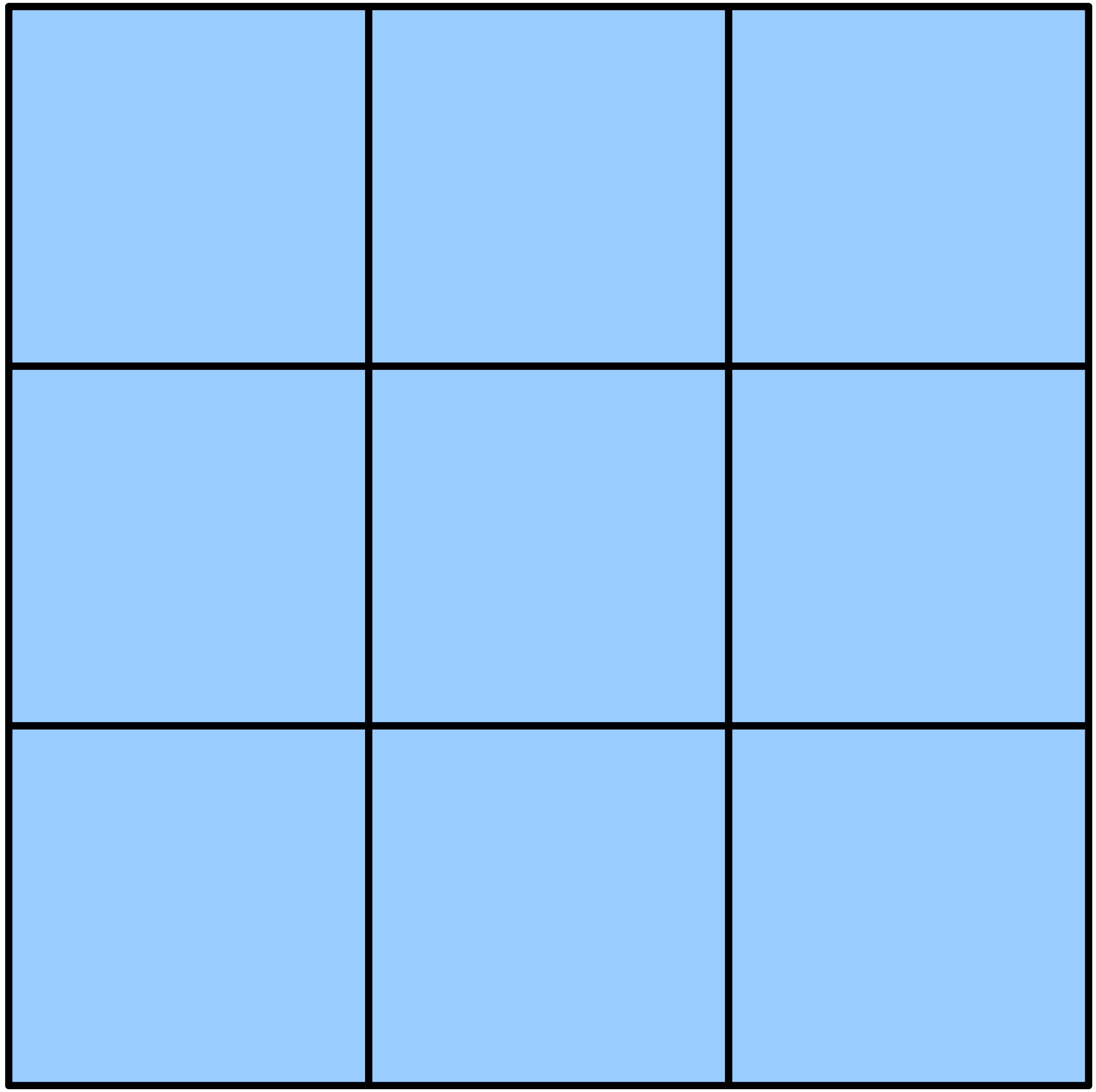


Subdividing a Square



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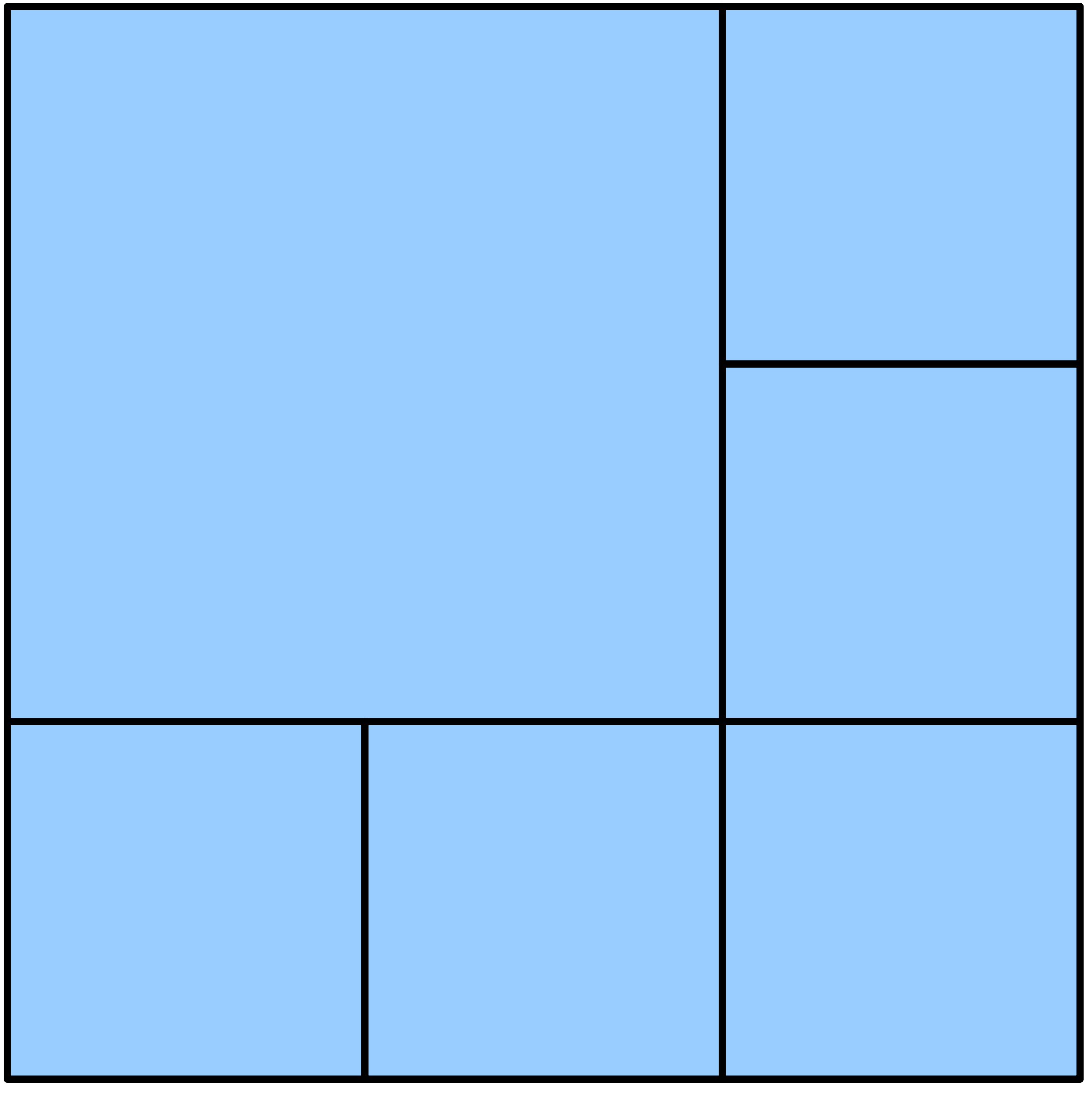

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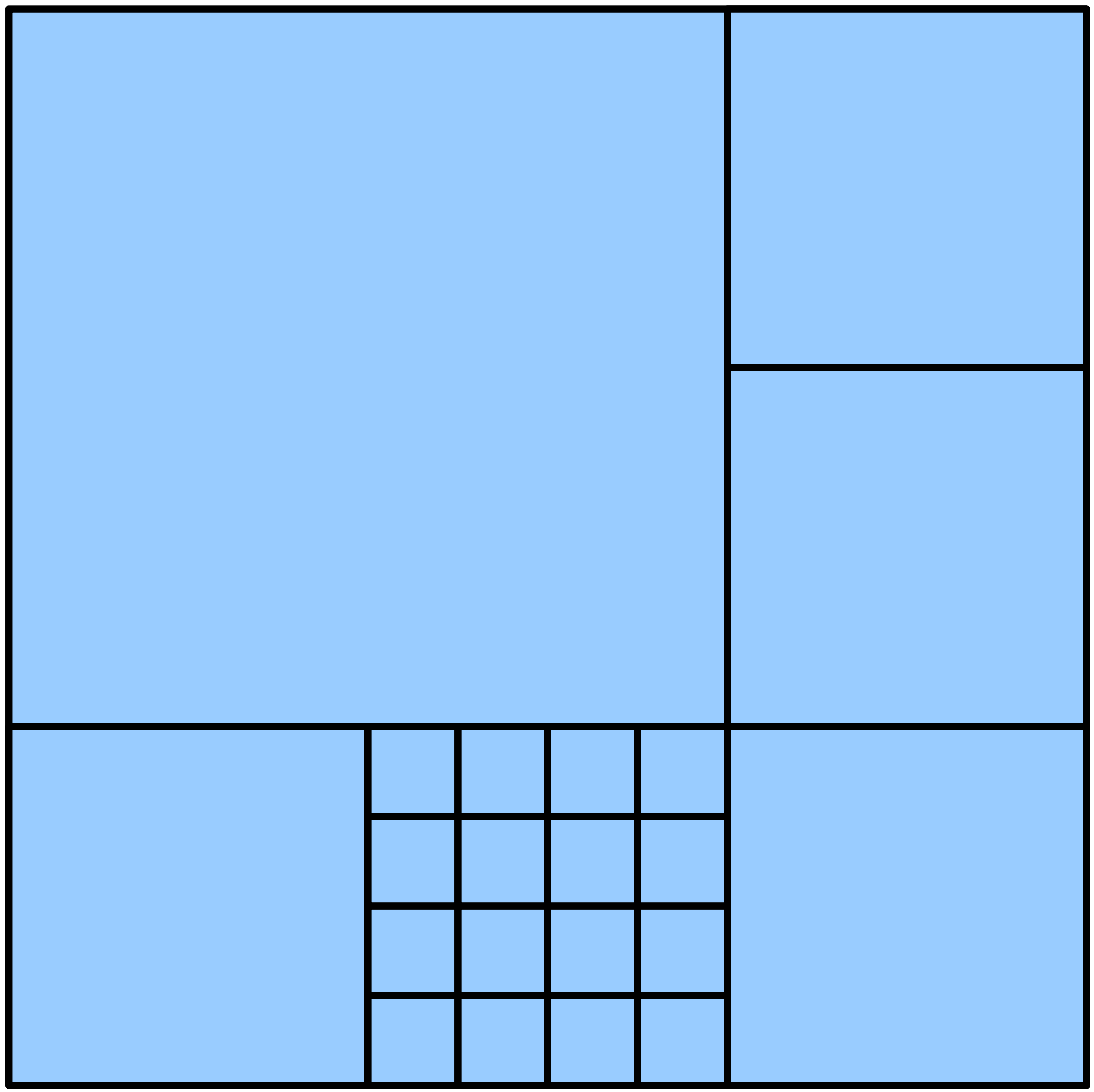

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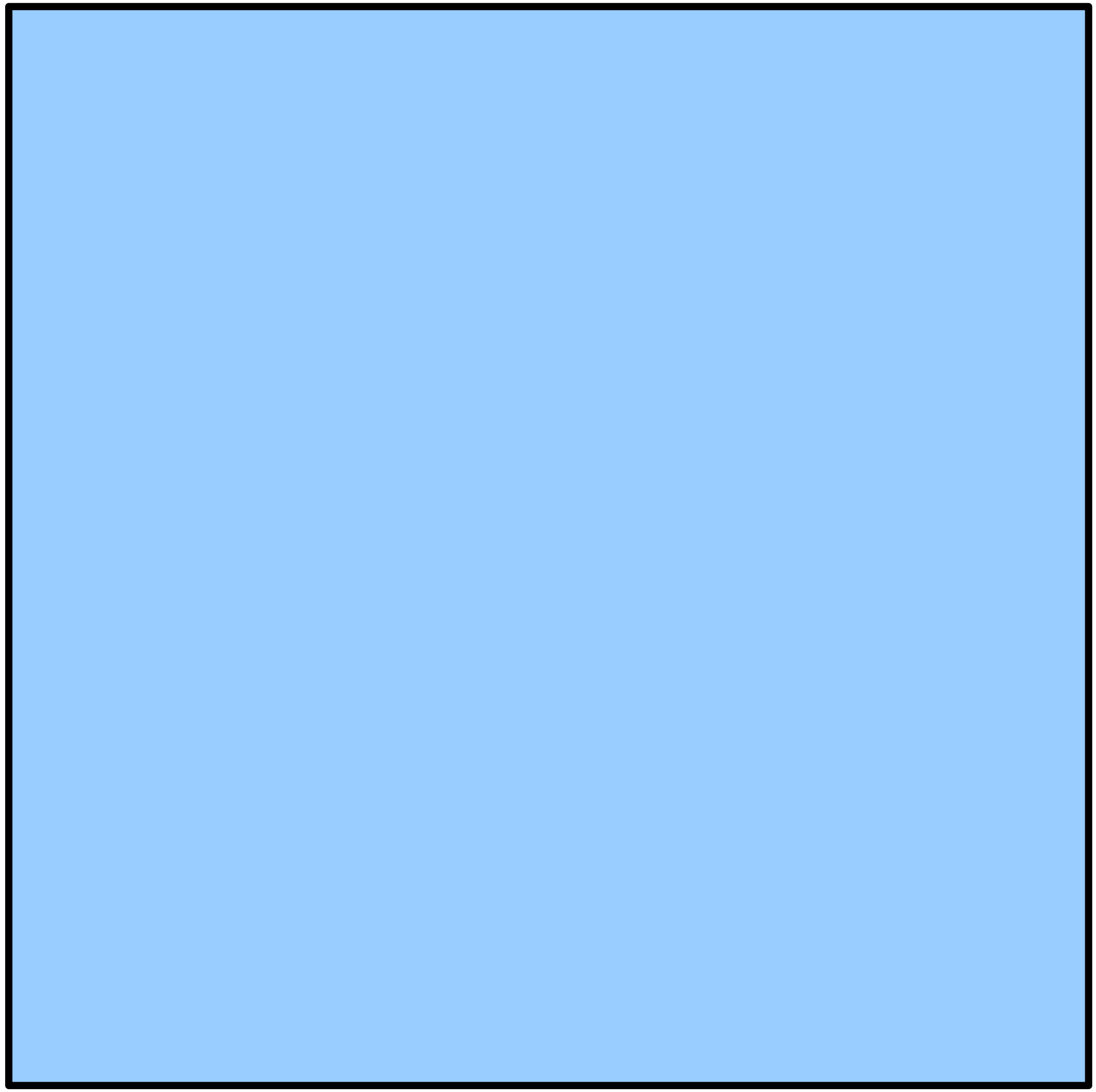

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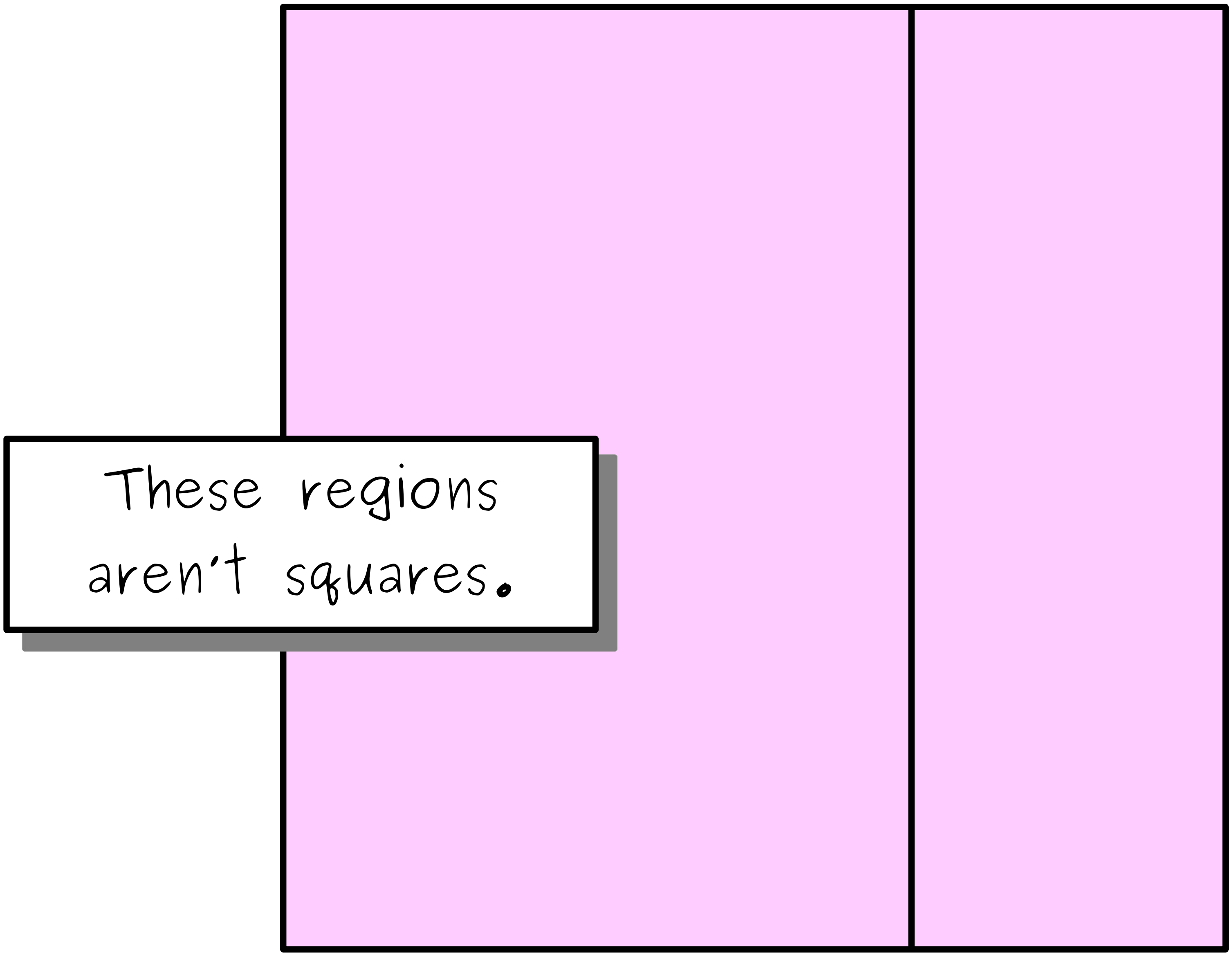

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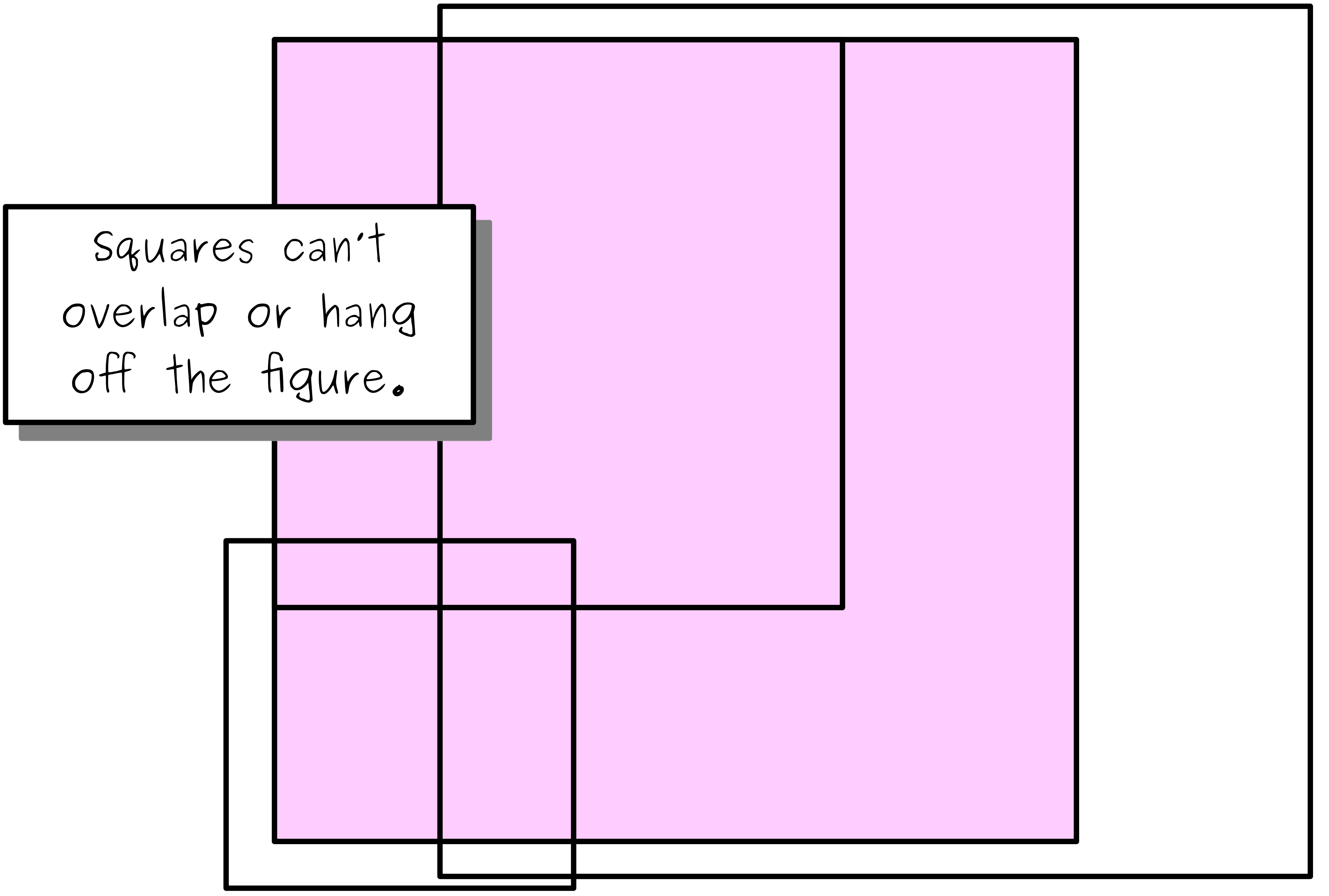

For what values of n can a square be subdivided into n squares?

Try out some numbers n from 1 to 12. Which values of n work?

Answer at **https://cs103.stanford.edu/pollev**

1 2 3 4 5 6 7 8 9 10 11 12

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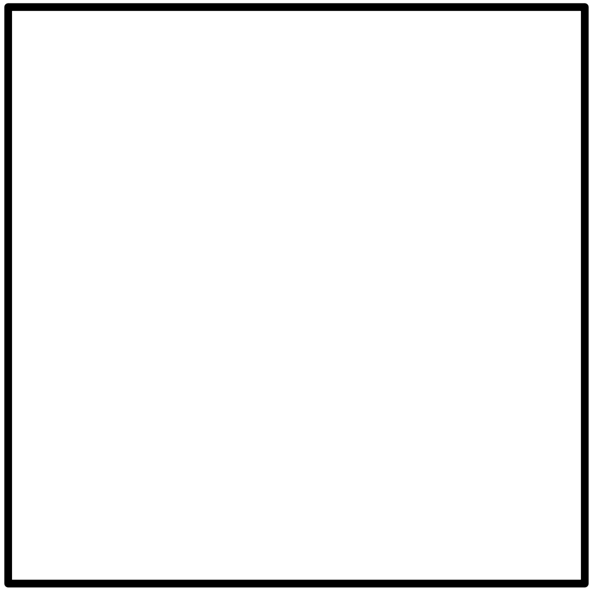
1

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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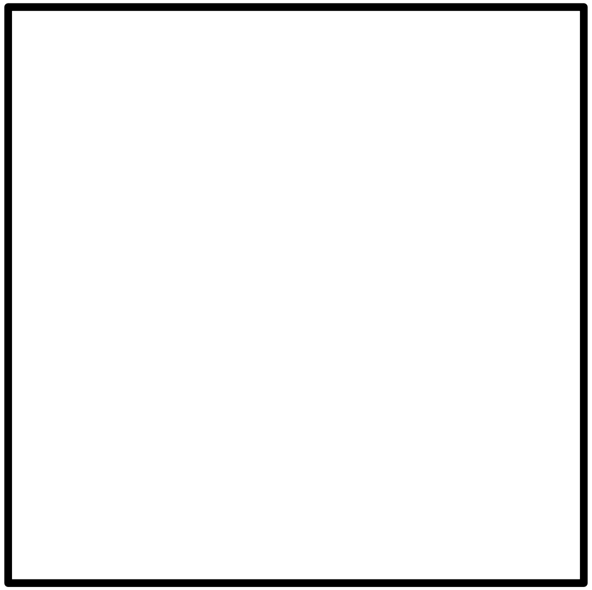
1	2
4	3

1 ~~2~~ ~~3~~ 4 5 6 7 8 9 10 11 12



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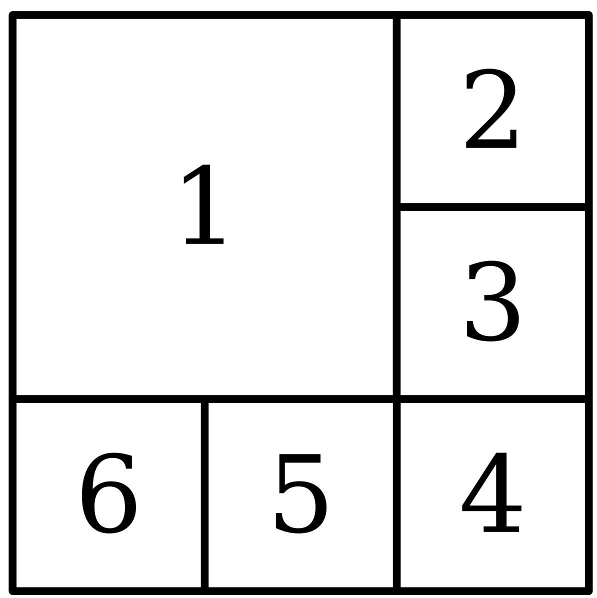

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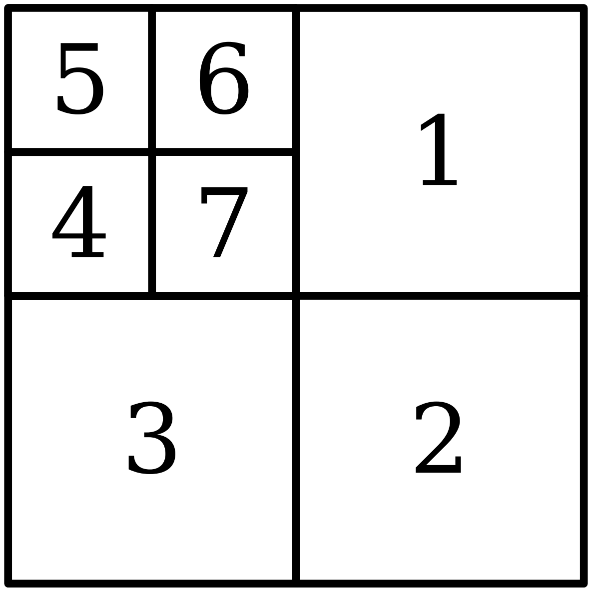

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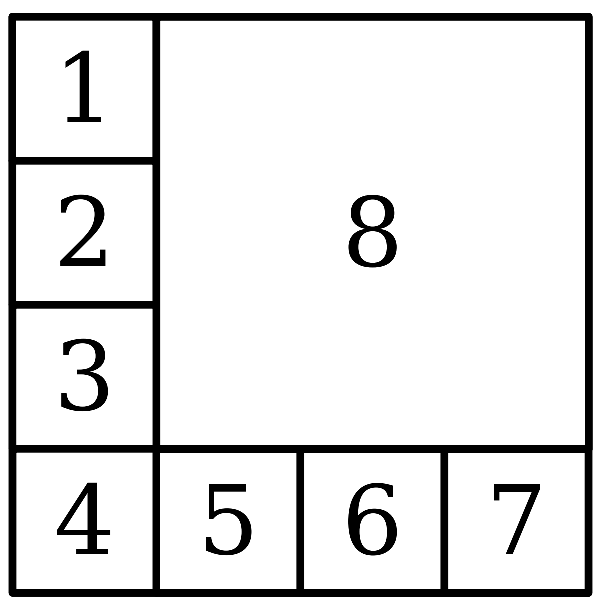
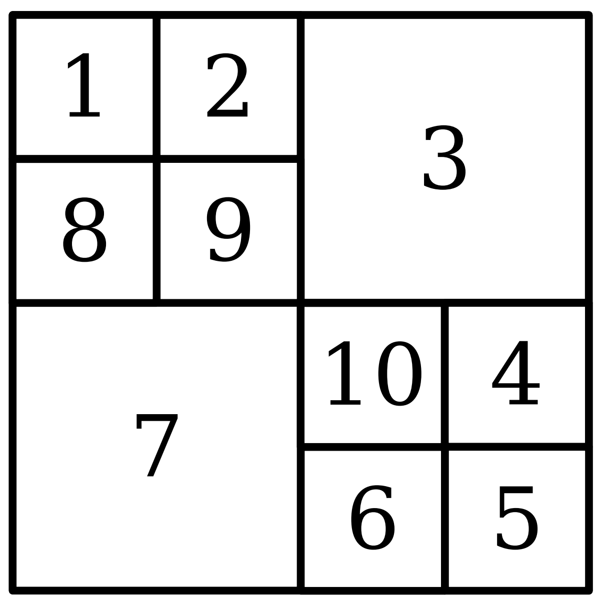


1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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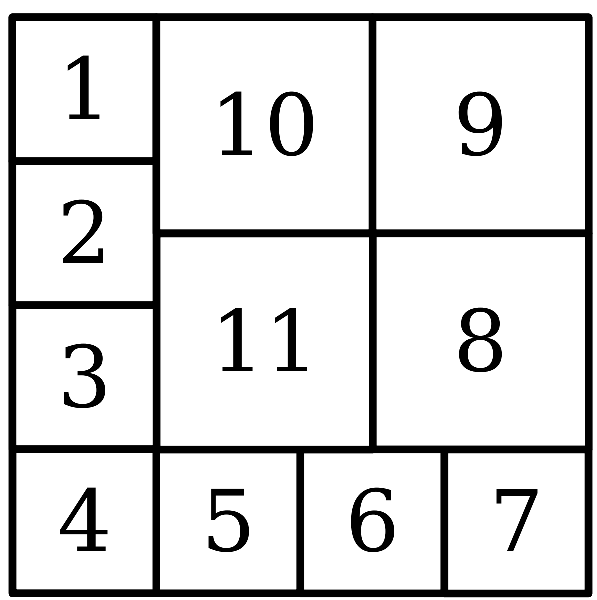

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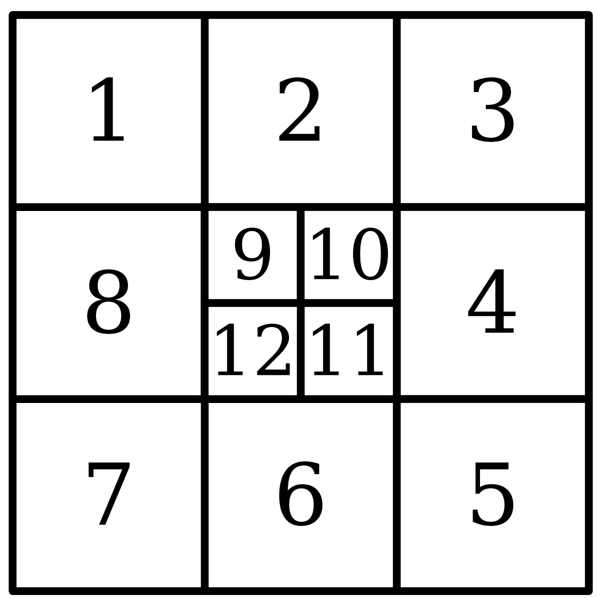

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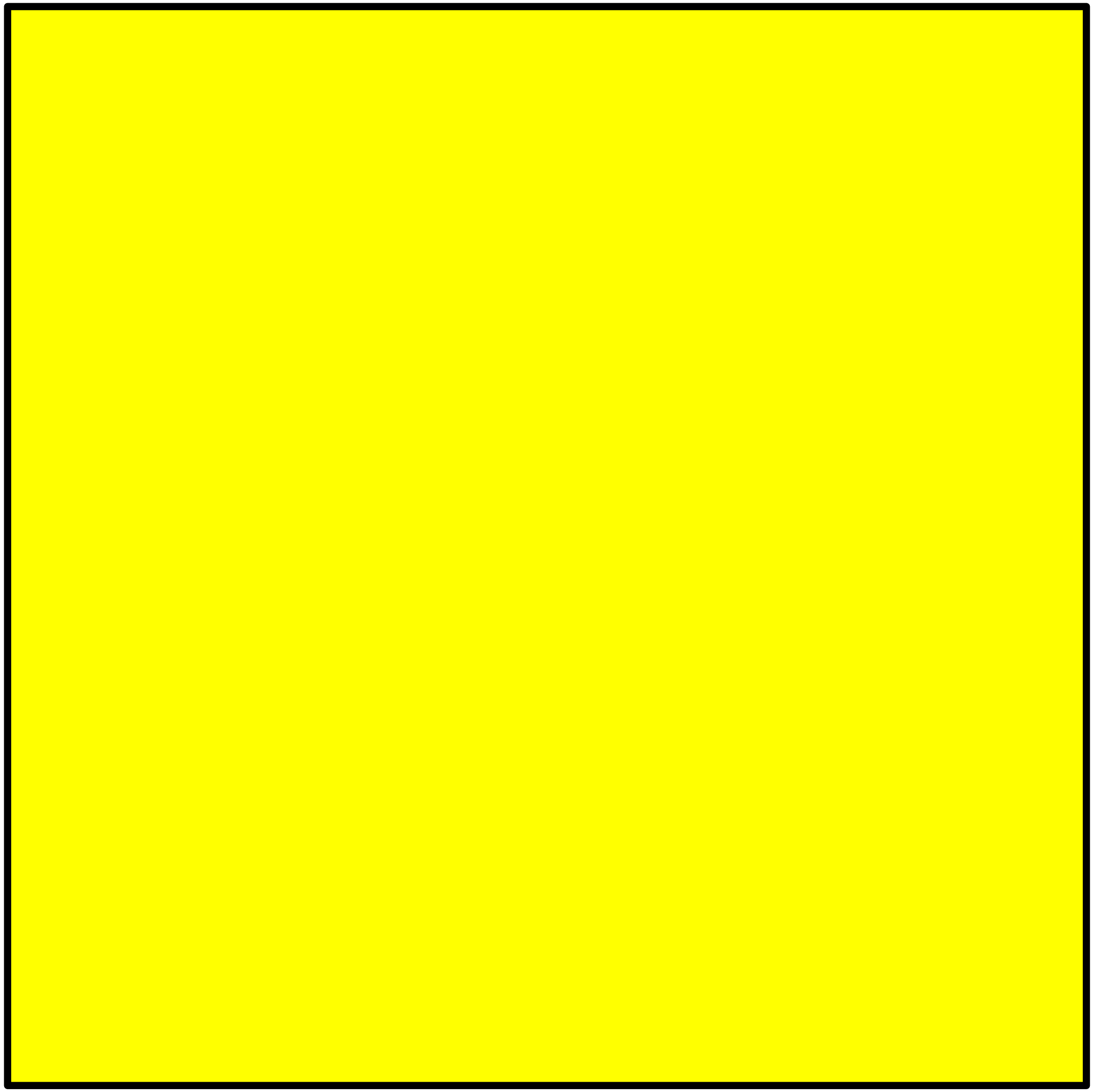

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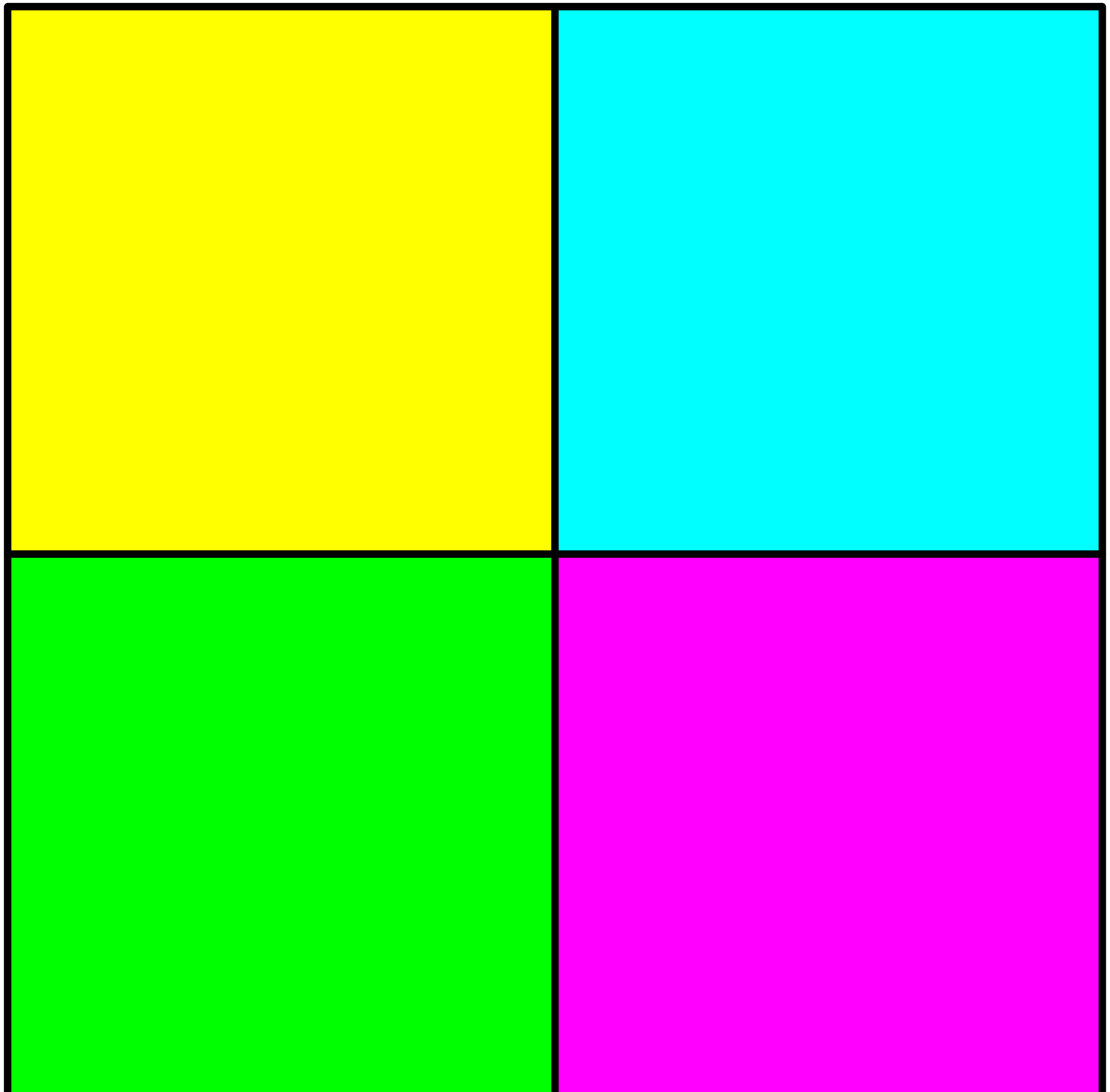

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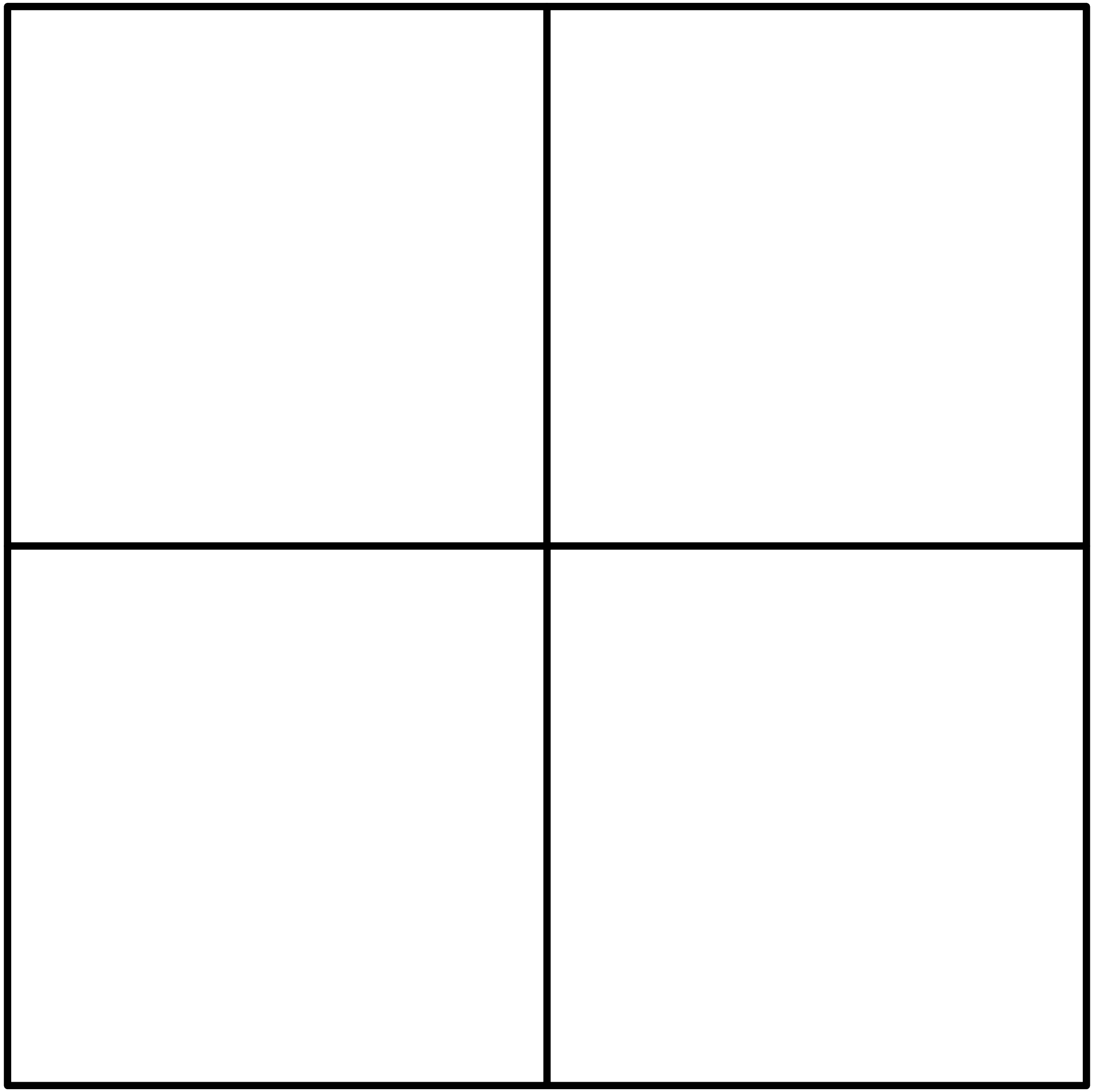

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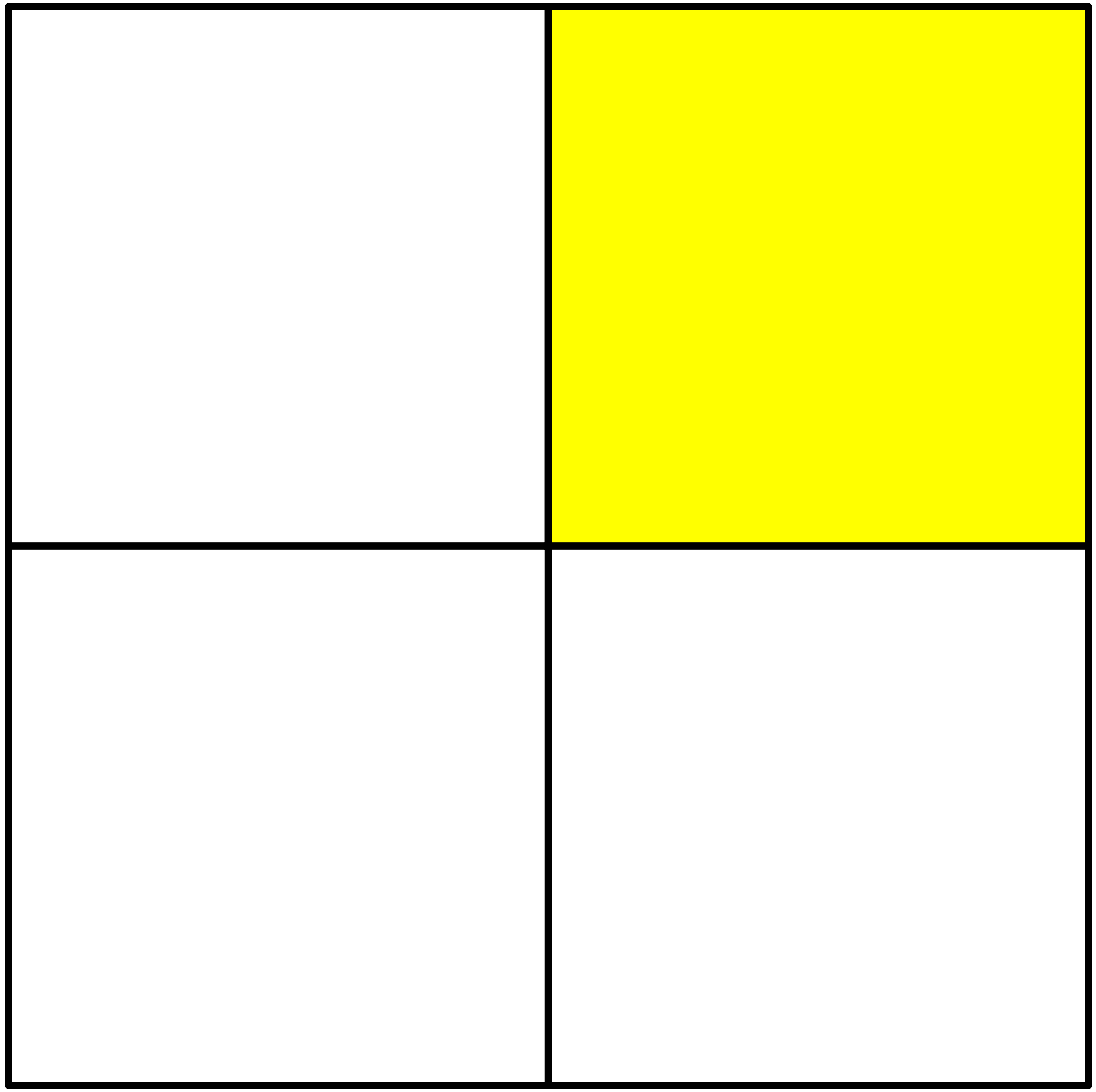

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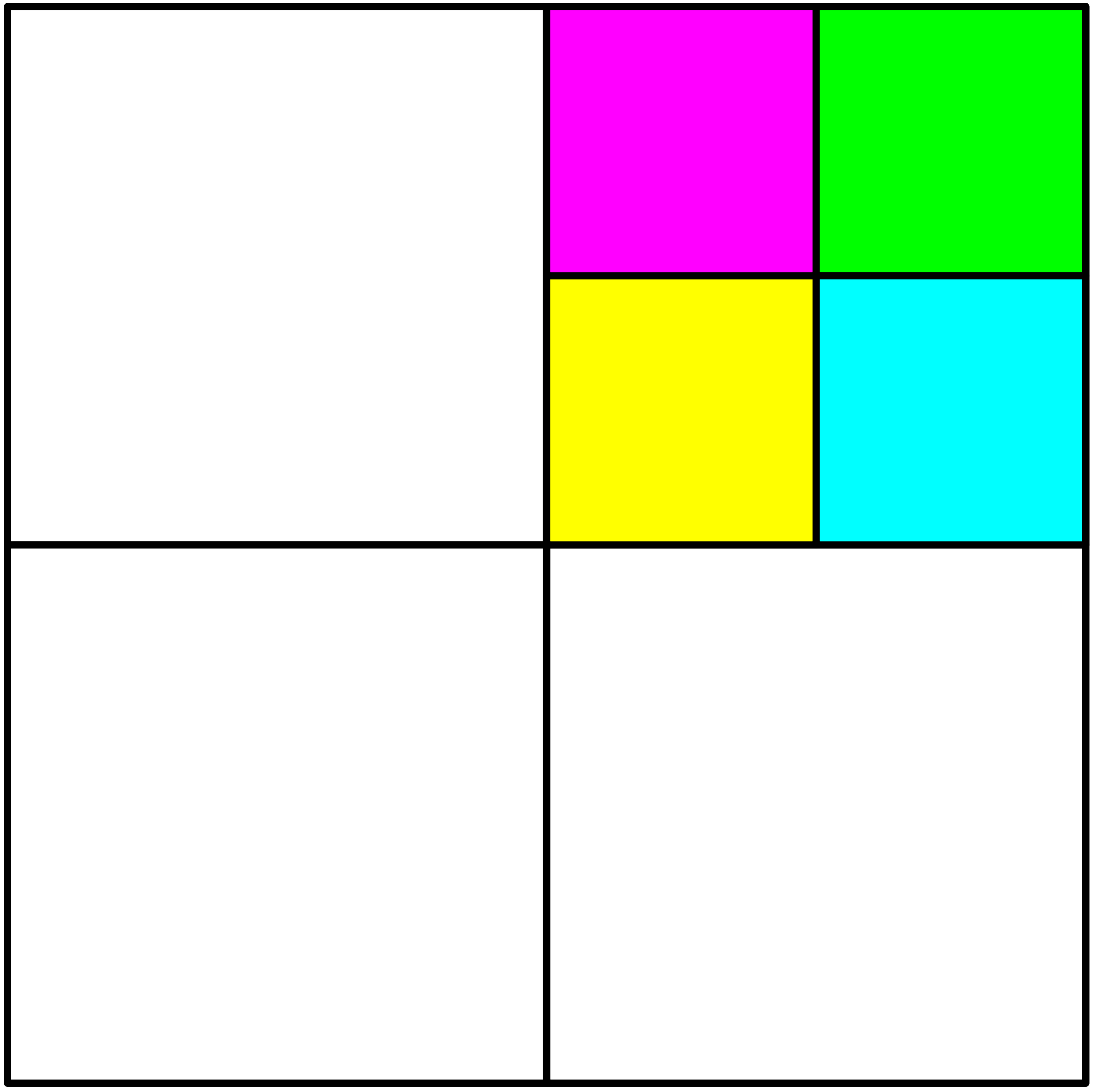

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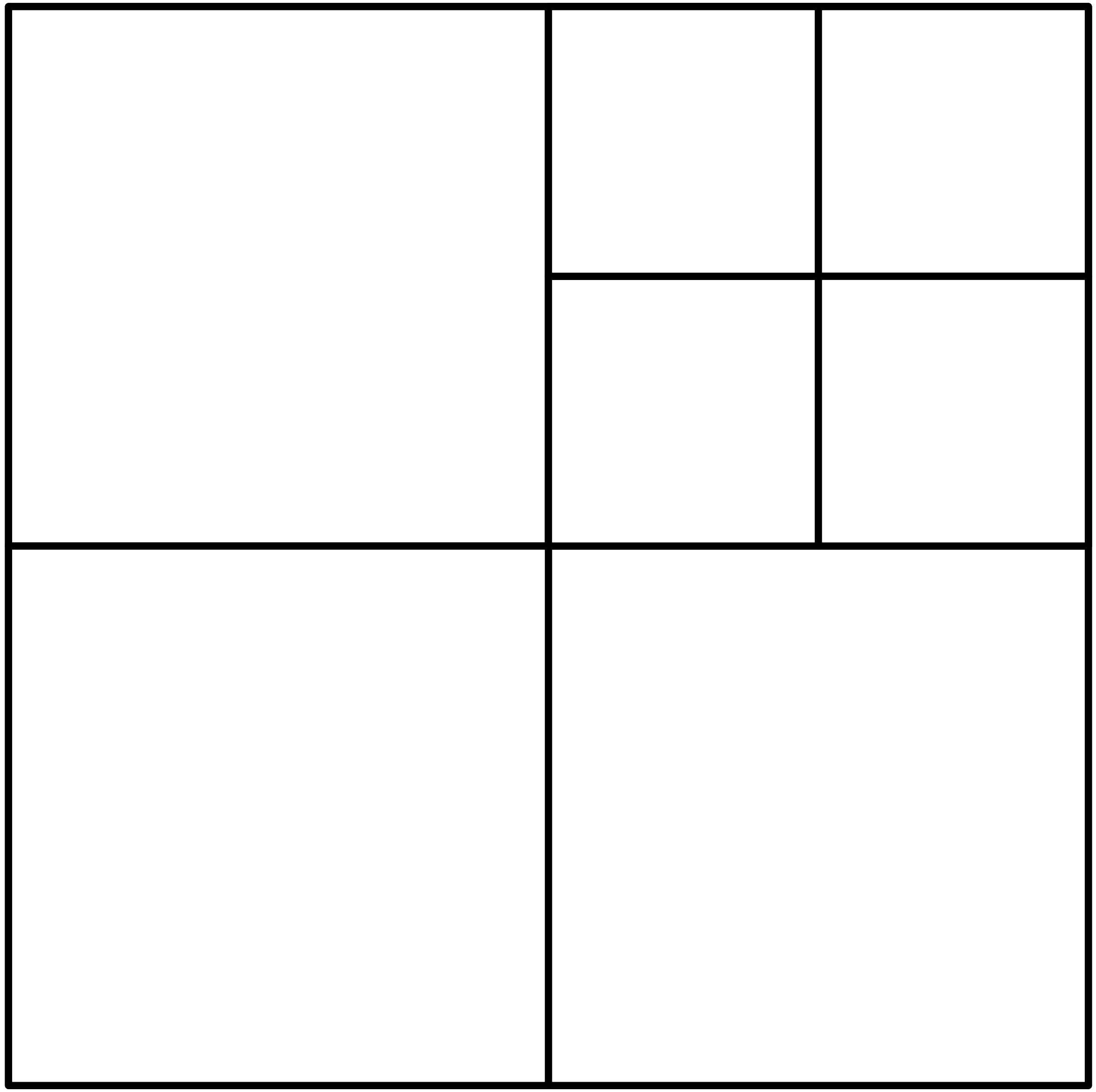

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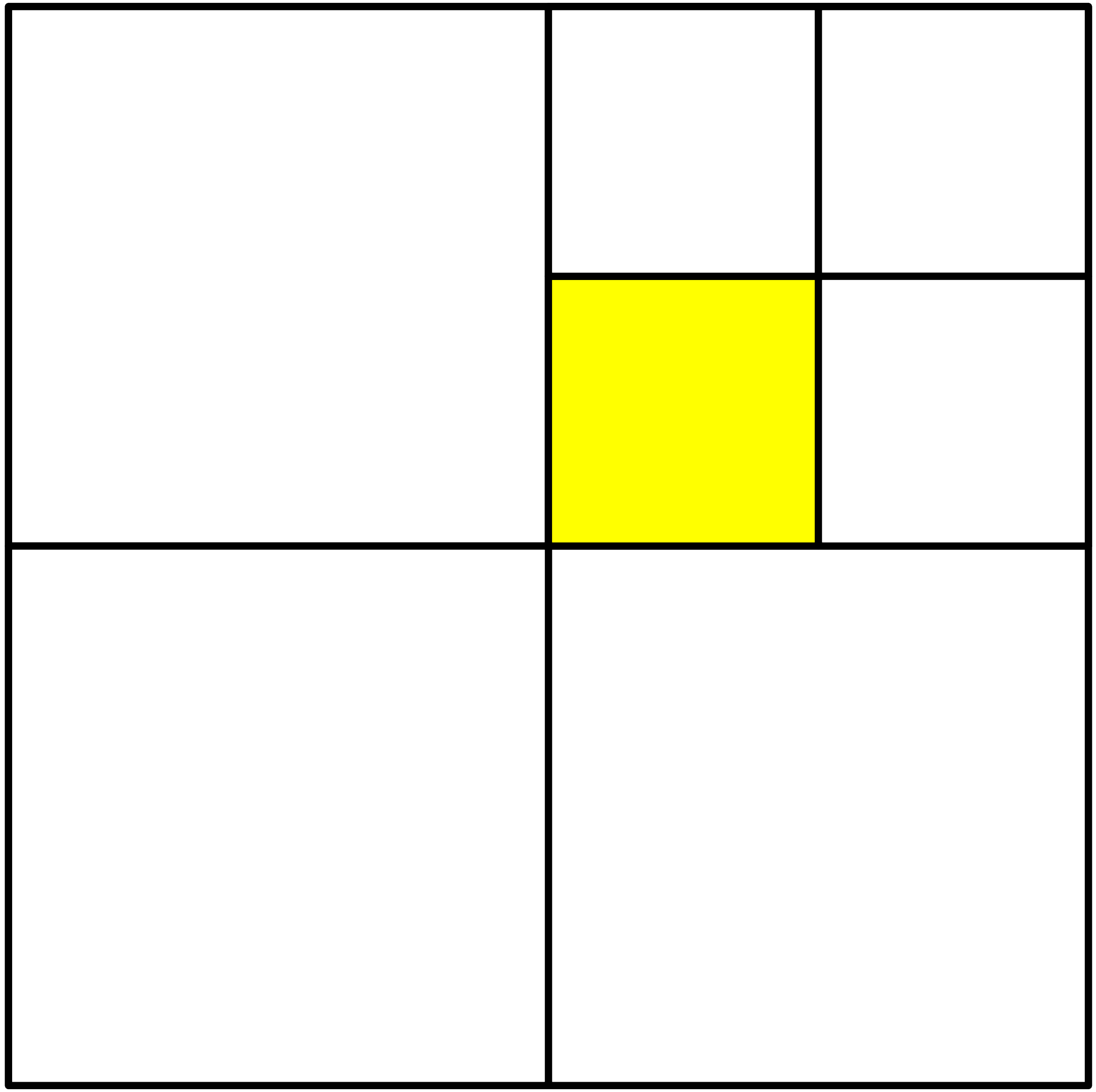

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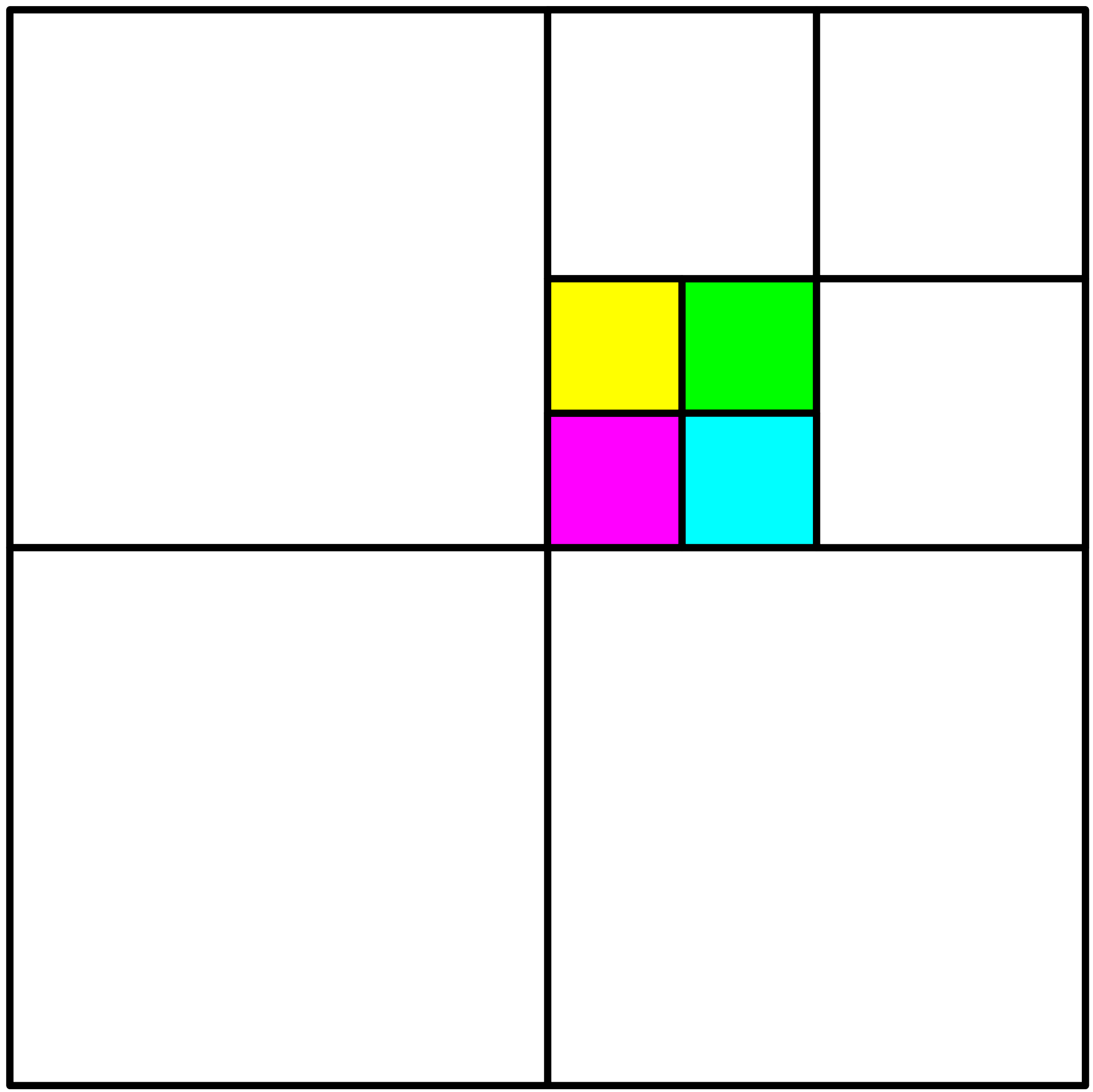

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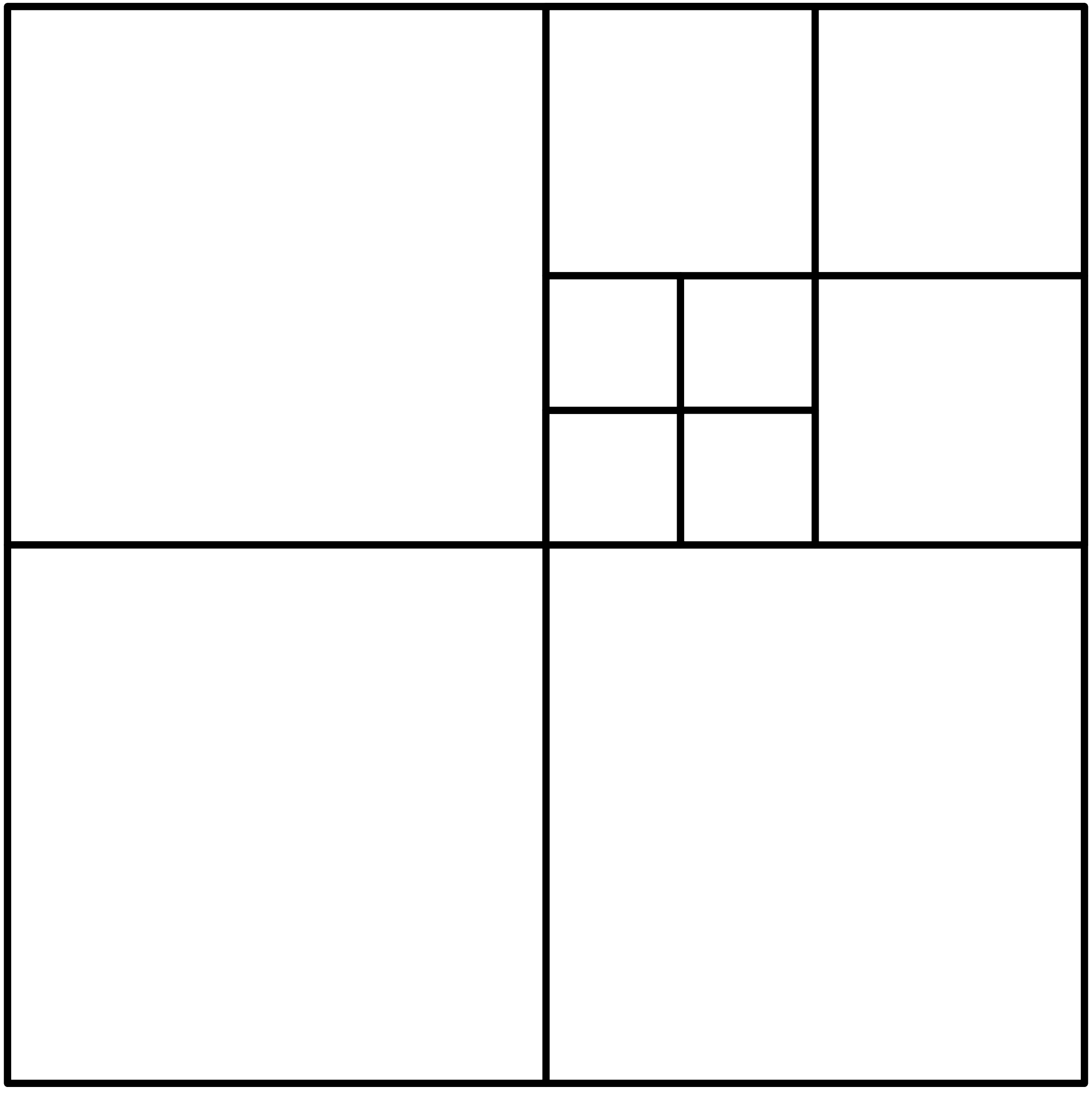

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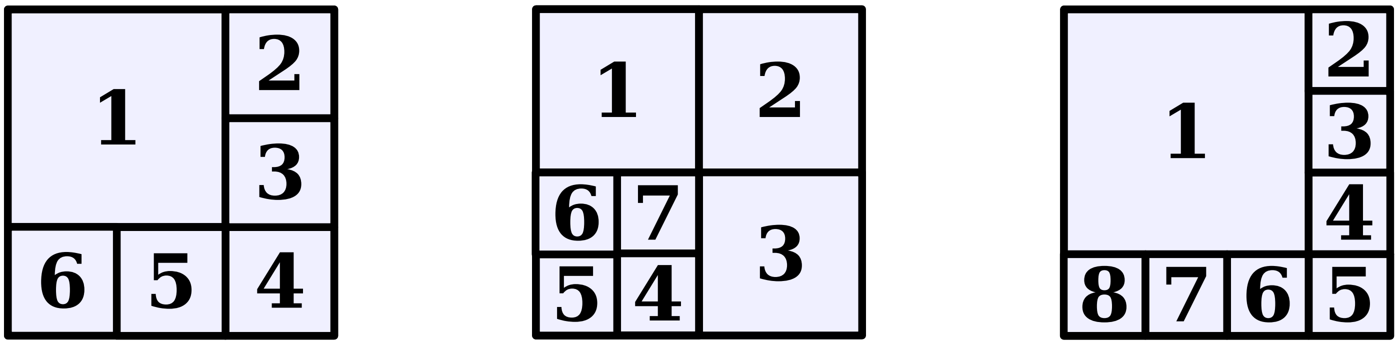

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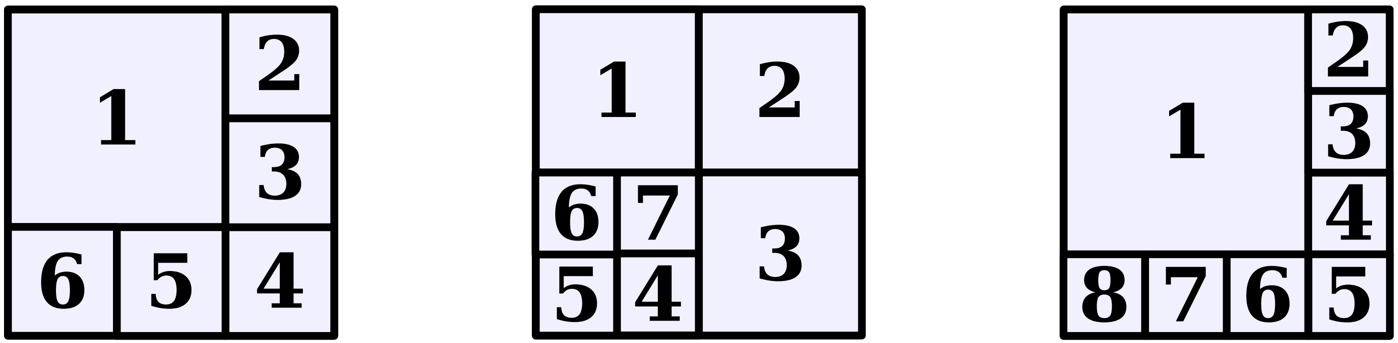

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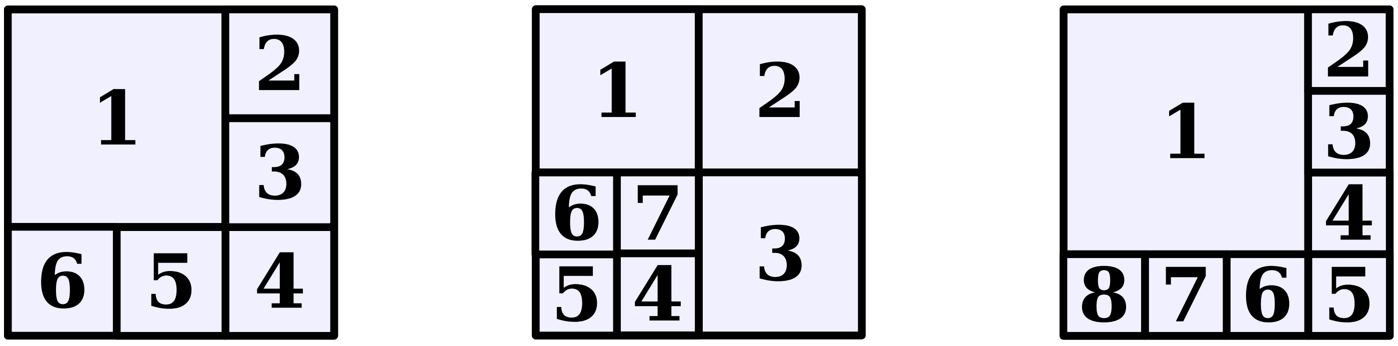

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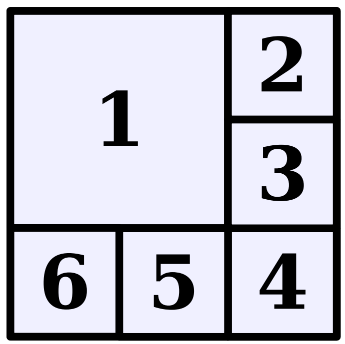

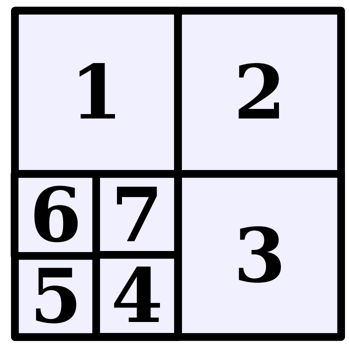

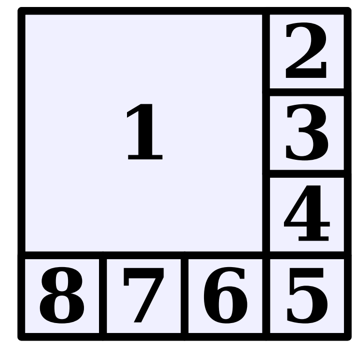

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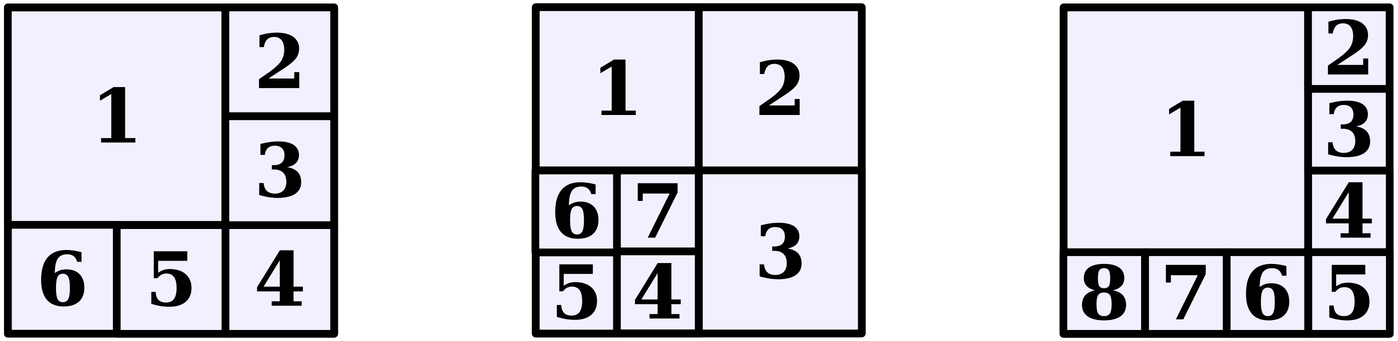

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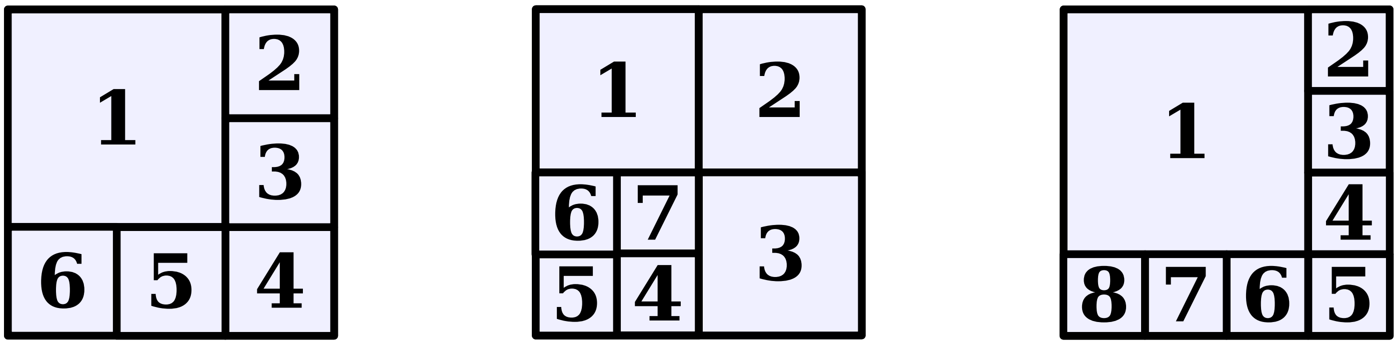

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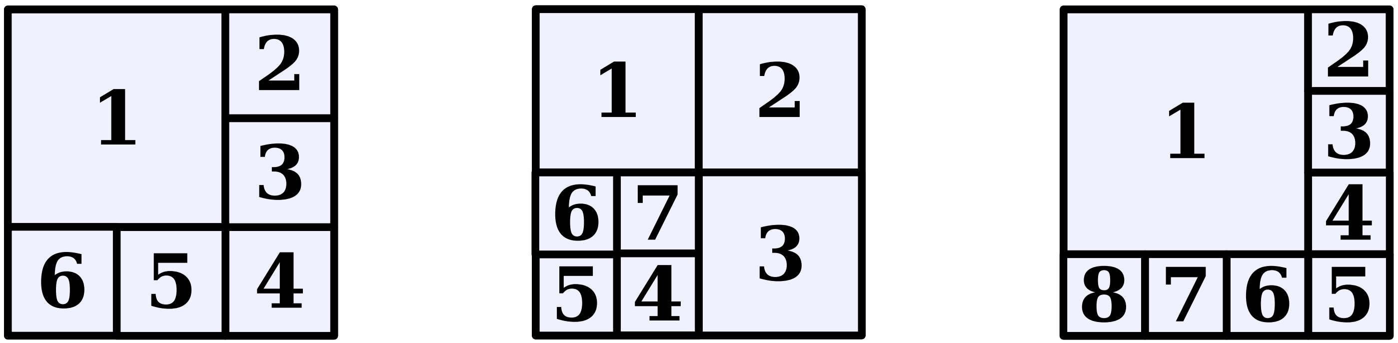

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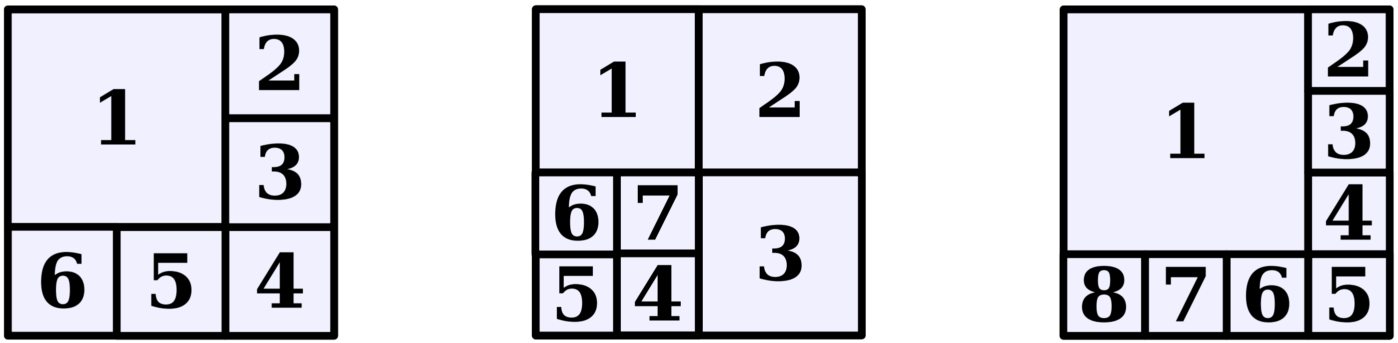

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Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you.

More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

Next Time

- ***"Build Up" vs "Build Down"***
  - A subtle but key point in induction proofs.
- ***Complete Induction***
  - Expanding our inductive hypothesis.